Vektoranalysis

Linienintegrale und Wegabhängigkeit. Beispiele

Änderung der kinetischen Energie:

$$T = \frac{m}{2} v^2 = \frac{m}{2} \vec{v} \cdot \vec{v}$$

$$\frac{dT}{dt} = \frac{m}{2}\left(\dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}}\right) = m \dot{\vec{v}} \cdot \vec{v} = \underbrace{m\vec{a}}_{\vec{F}} \cdot \vec{v}$$

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$, Bilanzgleichung

Leistung

$$\Delta T = \int_{t_1}^{t_2} \frac{dT}{dt}\, dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

– 1-dimensionale Bewegung

$$\int_{t_1}^{t_2} F \frac{dx}{dt}\, dt = \int_{x_1}^{x_2} F\, dx = W$$ Arbeit („Kraft × Weg")

$x_1 = x(t_1)$

$x_2 = x(t_2)$

$$\int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \underline{\int_C \vec{F} \cdot d\vec{r} = W}$$, Linienintegral

Kurven – " –

Weg – " –

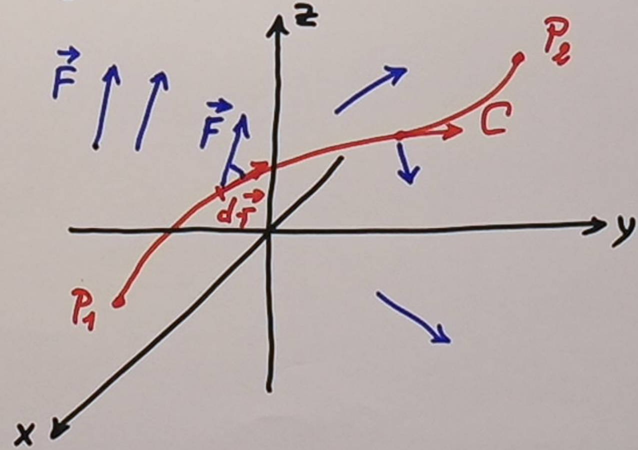
(Anmerkung: Lorentz-Kraft

$\vec{F} = q\, \vec{v} \times \vec{B} \rightarrow \vec{F} \cdot \vec{v} = 0$)

$$\vec{F} = F_1(x,y,z)\, \vec{i} + F_2(x,y,z)\, \vec{j} + F_3(x,y,z)\, \vec{k}$$

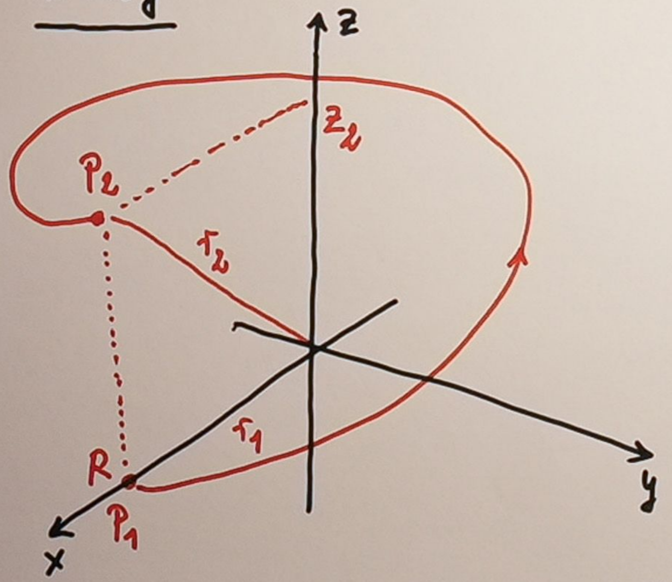
$$d\vec{r} = dx \cdot \vec{i} + dy \cdot \vec{j} + dz \cdot \vec{k}$$

$$W = \int_C (F_1 dx + F_2 dy + F_3 dz)$$



Beispiele: Zwei Wege

1. Weg



$$\vec{r}(t) = R\cos\omega t \cdot \vec{i} + R \cdot \sin\omega t \cdot \vec{j} + ct \cdot \vec{k}$$

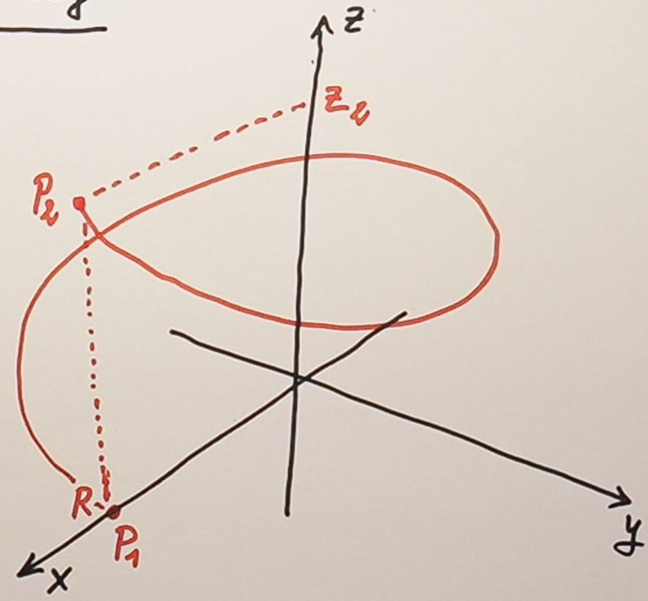
$$0 \le t \le \frac{2\pi}{\omega}, \quad c = const$$

P_1: $\vec{r}_1 = R\,\vec{i}$

P_2: $\vec{r}_2 = R\,\vec{i} + \frac{2\pi c}{\omega}\vec{k}$

$$\vec{v}(t) = \omega R\left(-\sin\omega t \cdot \vec{i} + \cos\omega t \cdot \vec{j}\right) + c\vec{k}$$

2. Weg



$$\vec{r}(t) = R\cos\omega t\,\vec{i} - R\sin\omega t \cdot \vec{j} + ct \cdot \vec{k}$$

$$\vec{v}(t) = -\omega R\left(\sin\omega t \cdot \vec{i} + \cos\omega t \cdot \vec{j}\right) + c\vec{k}$$

1. Kraftfeld

$$\vec{F} = \alpha(y\vec{i} - x\vec{j}) + f_0\vec{k}$$

1. Weg

$$\vec{F} = \alpha R(\sin\omega t \cdot \vec{i} - \cos\omega t \cdot \vec{j}) + f_0\vec{k}$$

$$\vec{F}\cdot\vec{v} = -\alpha\omega R^2 + c f_0 = \text{const}$$

Arbeit: $$W_1 = \int_0^{2\pi/\omega} \vec{F}\cdot\vec{v}\,dt = \frac{2\pi}{\omega}(-\alpha\omega R^2 + c f_0)$$

$$= \underline{-2\pi\alpha R^2 + f_0 \cdot z_2}$$

2. Weg

$$\vec{F} = -\alpha R(\sin\omega t \cdot \vec{i} + \cos\omega t \cdot \vec{j}) + f_0\vec{k}$$

$$\vec{F}\cdot\vec{v} = \alpha\omega R^2 + c f_0 = \text{const}$$

$$W_2 = \frac{2\pi}{\omega}(\alpha\omega R^2 + c f_0) = \underline{2\pi\alpha R^2 + f_0 \cdot z_2}$$

Resultat: $W_2 \neq W_1$, wegabhängig

Linienintegrale und Wegunabhängigkeit. Beispiele

Wege unverändert

2. Kraftfeld : Zentralkraftfeld

$$\vec{F} = \frac{\alpha}{r^2} \cdot \frac{\vec{r}}{r}$$

Gravitostatik
Elektrostatik

Parametrisierung: 1. Weg

$$\vec{F} = \alpha \frac{R\cos\omega t \cdot \vec{i} + R\sin\omega t \cdot \vec{j} + ct \cdot \vec{k}}{(R^2 + c^2t^2)^{3/2}}$$

$$\vec{F} \cdot \vec{v} = \frac{\alpha c^2 t}{(R^2 + c^2t^2)^{3/2}}$$

$$W_1 = \int_0^{2\pi/\omega} \vec{F} \cdot \vec{v}\, dt = -\alpha \frac{1}{\sqrt{R^2 + c^2t^2}}\Bigg|_0^{2\pi/\omega} = \alpha\left(\frac{1}{R} - \frac{1}{\sqrt{R^2 + \frac{4\pi^2c^2}{\omega^2}}}\right)$$

$$= \alpha\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Koordinaten des Anfangs- und Endpunktes

2. Weg

$$\vec{F} = \alpha \frac{R\cos\omega t \cdot \vec{i} - R\sin\omega t \cdot \vec{j} + ct \cdot \vec{k}}{(R^2 + c^2t^2)^{3/2}}$$

$$\vec{F} \cdot \vec{v} = \frac{\alpha c^2 t}{(R^2 + c^2t^2)^{3/2}}$$

$\underline{\underline{W_1 = W_2}}$, wegunabhängig

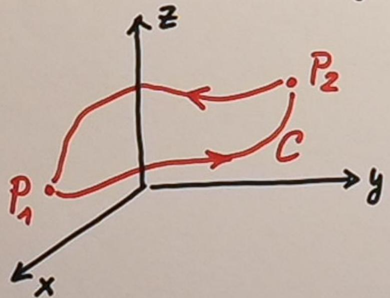
Ein differentielles Kriterium für Wegunabhängigkeit

Versuch: $\vec{F}\cdot\vec{v} = -\dfrac{dU}{dt}$, $U = U(x,y,z)$

Arbeit: $$W = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = -\int_{t_1}^{t_2} \frac{dU}{dt}\,dt$$
$$= -\int_{P_1}^{P_2} dU = -(U_2 - U_1)$$
$$U_1 = U(P_1)$$
$$U_2 = U(P_2)$$

geschlossener Weg:

$$W = -\oint_C dU$$
$$= -\left(\int_{P_1}^{P_2} dU + \int_{P_2}^{P_1} dU\right) = 0$$



$$\frac{dU}{dt} = \vec{v}\cdot \operatorname{grad} U$$ „Reisegleichung"
$$-\frac{dU}{dt} = \vec{F}\cdot\vec{v}$$

Vergleich: $\vec{F} = -\operatorname{grad} U$

$\downarrow$

$$F_1 = -\frac{\partial U}{\partial x}, \quad F_2 = -\frac{\partial U}{\partial y}, \quad F_3 = -\frac{\partial U}{\partial z}$$

$$\rightarrow \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

Anmerkung: $W = T_2 - T_1 = -(U_2 - U_1)$
$$T_1 + U_1 = T_2 + U_2$$

$\vec{F}$: konservativ

U : potentielle Energie (Potential)

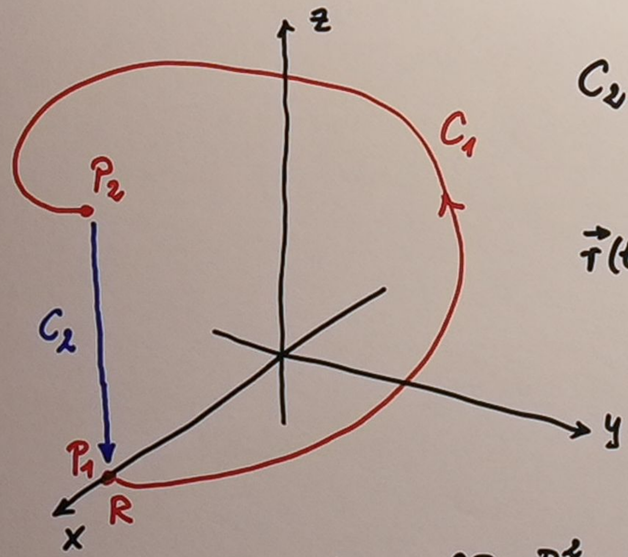
Beispiele:

1. Kraftfeld: $F_1 = \alpha y\,,\quad F_2 = -\alpha x\,,\quad F_3 = f_0$

$$\frac{\partial F_1}{\partial y} = \alpha\,,\quad \frac{\partial F_2}{\partial x} = -\alpha$$

nicht konservativ

Arbeit auf geschlossenem Weg:



C_2: $\vec{v} = c\vec{k}$

$\vec{F}\vec{v} = cf_0$

$\vec{r}(t) = R\vec{i} + ct\vec{k}$

C_1: $W_1 = -2\pi\alpha R^2 + f_0 z_2$

C_2: $W_2 = \int_{2\pi/\omega}^{0} \vec{F}\vec{v}\,dt = -cf_0\frac{2\pi}{\omega} = -f_0 z_2$

$$W = \oint_{C_1+C_2} \vec{F}\,d\vec{r} = -2\pi\alpha R^2 \neq 0$$

2. Kraftfeld: $F_1 = \alpha\frac{x}{r^3}\,,\quad F_2 = \alpha\frac{y}{r^3}\,,\quad F_3 = \alpha\frac{z}{r^3}$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial F_1}{\partial y} = -3\alpha\frac{xy}{r^5}\,,\quad \frac{\partial F_2}{\partial x} = -3\alpha\frac{xy}{r^5} \quad \text{usw.}$$

Konservativ

C_1: $W_1 = \alpha\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

C_2: $\vec{F} = \alpha\frac{R\vec{i} + ct\vec{k}}{(R^2+c^2t^2)^{3/2}}$

$$\vec{F}\vec{v} = \frac{\alpha c^2 t}{(R^2+c^2t^2)^{3/2}}$$

$$W_2 = \int_{2\pi/\omega}^{0} \vec{F}\cdot\vec{v}\,dt = -W_1$$

$$W = W_1 + W_2 = 0$$

Die Berechnung des Potentials

Beispiel: Zentralkraftfeld

$$\vec{F} = \frac{\alpha}{r^2} \cdot \frac{\vec{r}}{r}$$

$$W = \alpha \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = U_1 - U_2 \,, \quad U_1 \equiv U(P_1) = \frac{\alpha}{r_1}$$

$$U_2 \equiv U(P_2) = \frac{\alpha}{r_2}$$

Potential: $U = \frac{\alpha}{r} + U_0 \,, \quad U_0 = \text{const}$

$\tilde{U} = U + U_0$: $\vec{\tilde{F}} = -\text{grad}\,\tilde{U} = -\text{grad}(U + U_0)$

$$= -\text{grad}\,U = \vec{F}$$

$U(\infty) \overset{!}{=} 0 \rightarrow U_0 = 0 \,, \quad U = \frac{\alpha}{r}$ (Eichung)

Potentialberechnung: 1. Verfahren

direkte Integration

$$\frac{\partial U}{\partial x} = -F_1 = -\alpha \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$U = -\alpha \int \frac{x\,dx}{(x^2+y^2+z^2)^{3/2}}$$

$$= \alpha \cdot \frac{1}{(x^2+y^2+z^2)^{1/2}} + f(y,z)$$

$\longrightarrow \frac{\partial U}{\partial y} = -\alpha \frac{y}{r^3} + \frac{\partial f}{\partial y}$

andererseits: $\frac{\partial U}{\partial y} = -\alpha \frac{y}{r^3}$

$\Big\} \; \frac{\partial f}{\partial y} = 0 \,, \quad f = f(z)$

$\longrightarrow \frac{\partial U}{\partial z} = -\alpha \frac{z}{r^3} + \frac{df}{dz}$

andererseits: $\frac{\partial U}{\partial z} = -\alpha \frac{z}{r^3}$

$\Big\} \; \frac{df}{dz} = 0 \,, \quad f = \text{const} = U_0$

$$\underline{U = \frac{\alpha}{r} + U_0}$$

Potentialberechnung, 2. Verfahren

Linienintegral, spezieller Integrationsweg

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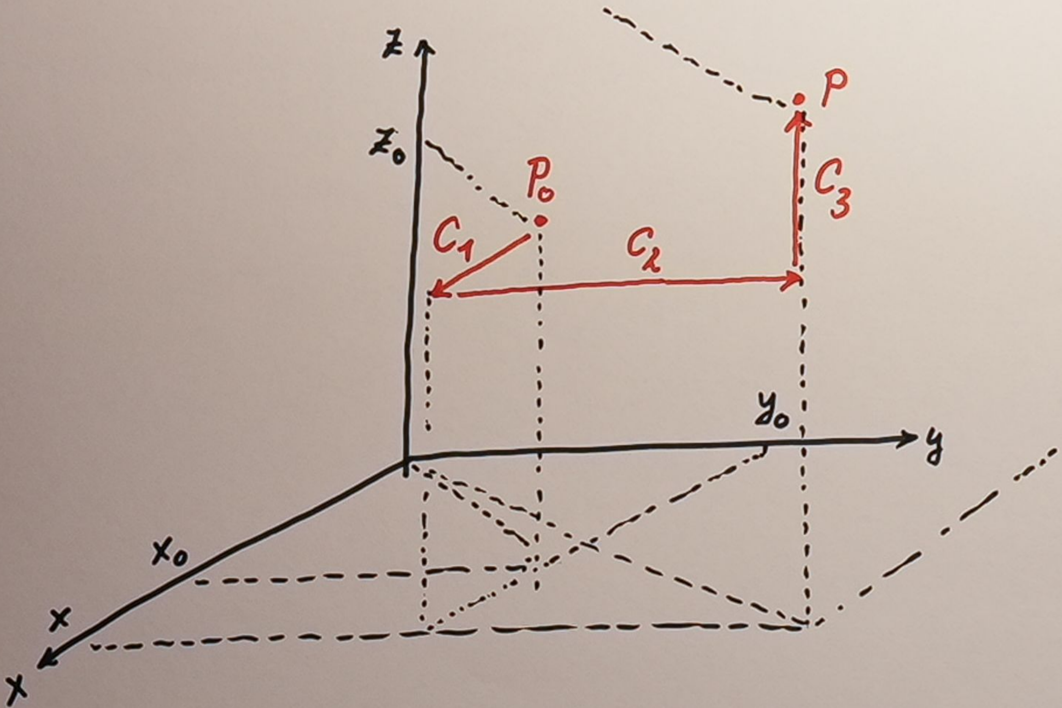
Wegstücke parallel zu Koordinatenachsen



C_1: $y = y_0$, $z = z_0$, $dy = 0 = dz$

$$\vec{F}d\vec{r} = F_1(x, y_0, z_0)dx$$

C_2: $x = \text{const}$, $z = z_0$, $dx = 0 = dz$

$$\vec{F}d\vec{r} = F_2(x, y, z_0)dy$$

C_3: $x = \text{const}$, $y = \text{const}$, $dx = 0 = dy$

$$\vec{F}d\vec{r} = F_3(x, y, z)dz$$

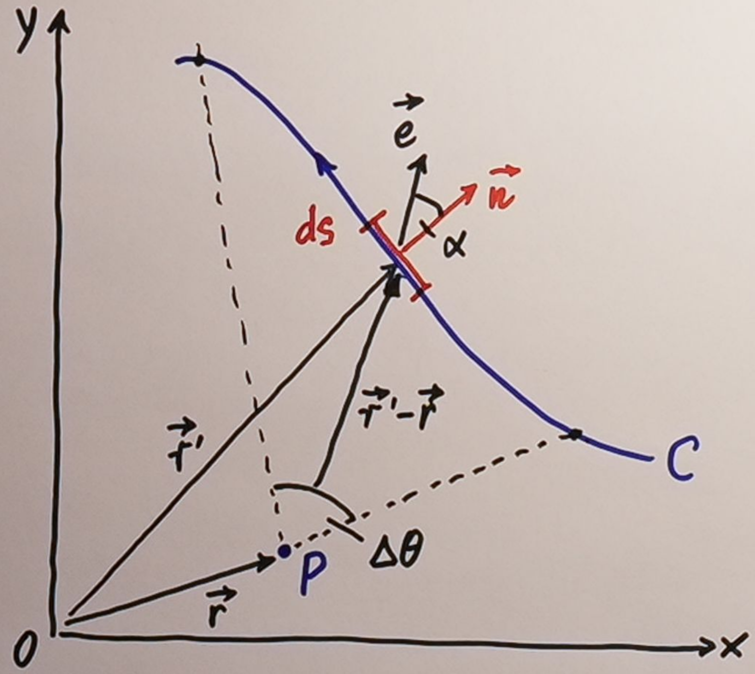
$$U = \int_C dU = -\int_C \vec{F}d\vec{r} = -\int_{C_1}\vec{F}d\vec{r} - \int_{C_2}\vec{F}d\vec{r} - \int_{C_3}\vec{F}d\vec{r}$$

$$U = -\int_{x_0}^{x} F_1(\xi, y_0, z_0)\,d\xi - \int_{y_0}^{y} F_2(x, \eta, z_0)\,d\eta - \int_{z_0}^{z} F_3(x, y, \zeta)\,d\zeta$$

Beispiel: Zentralkraftfeld

$$U = -\int_{x_0}^{x} \frac{\alpha\,\xi\,d\xi}{(\xi^2 + y_0^2 + z_0^2)^{3/2}} - \int_{y_0}^{y} \frac{\alpha\,\eta\,d\eta}{(x^2 + \eta^2 + z_0^2)^{3/2}} - \int_{z_0}^{z} \frac{\alpha\,\zeta\,d\zeta}{(x^2 + y^2 + \zeta^2)^{3/2}}$$

Der ebene Winkel als Linienintegral



„Blickrichtung"

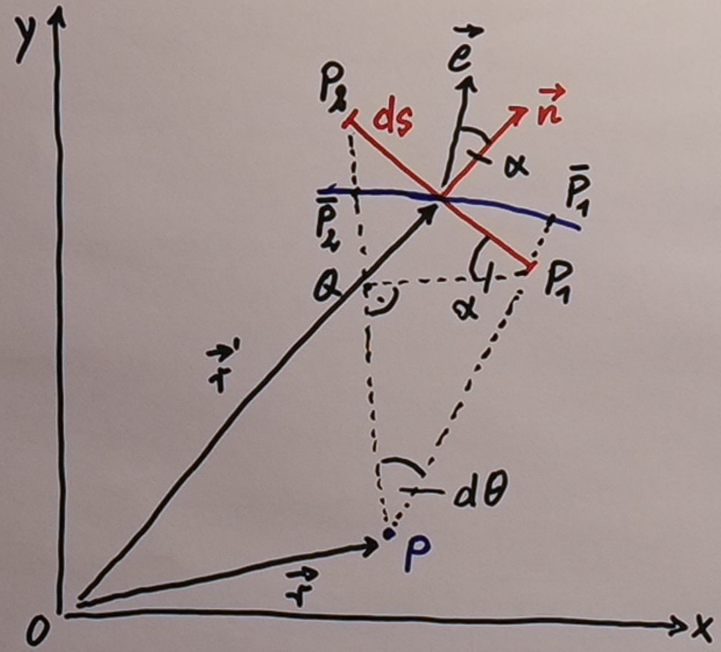
$$\vec{e} = \frac{\vec{r}'-\vec{r}}{|\vec{r}'-\vec{r}|}$$

$$\cos\alpha = \vec{e}\cdot\vec{n}$$

$$\overline{P_1Q} \approx \overset{\frown}{\overline{P}_1\overline{P}_2}$$

$$\Delta\theta = \int_C \frac{\vec{e}\cdot\vec{n}}{|\vec{r}'-\vec{r}|}\,ds$$

$$= \int_C \frac{(\vec{r}'-\vec{r})\cdot\vec{n}}{|\vec{r}'-\vec{r}|^2}\,ds$$



$$\overline{P_1Q} = ds\cdot\cos\alpha$$

$$= ds\cdot(\vec{e}\cdot\vec{n})$$

Bogenmaß:

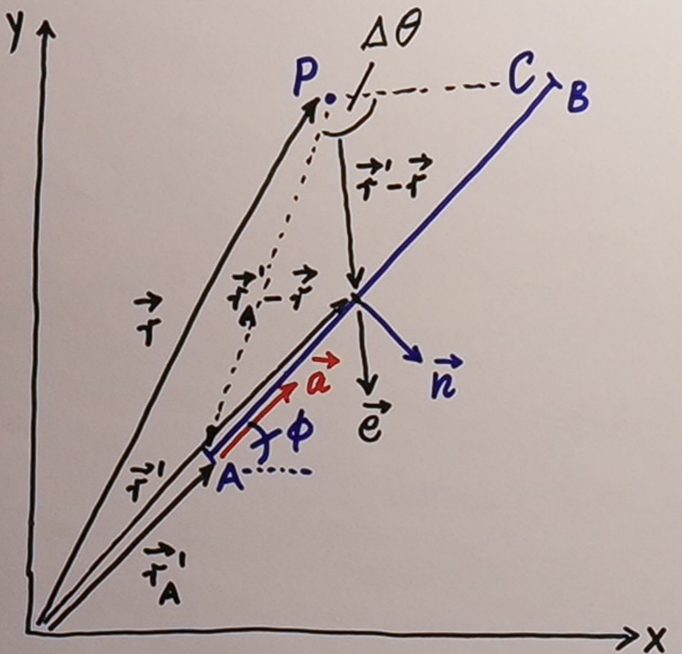
$$d\theta = \frac{\overset{\frown}{\overline{P}_1\overline{P}_2}}{|\vec{r}'-\vec{r}|}$$

$$d\theta = \frac{\overline{P_1Q}}{|\vec{r}'-\vec{r}|}$$

$$= \frac{\vec{e}\cdot\vec{n}}{|\vec{r}'-\vec{r}|}\,ds$$

Beispiel 1

$$\Delta\theta = \int_C \frac{\vec{e}\cdot\vec{n}}{|\vec{r}\,'-\vec{r}|}\,ds = \int_C \frac{(\vec{r}\,'-\vec{r})\cdot\vec{n}}{|\vec{r}\,'-\vec{r}|^2}\,ds$$



$\vec{a} = \cos\phi\cdot\vec{i} + \sin\phi\cdot\vec{j}$

$|\vec{a}| = 1$

$\vec{n} = \sin\phi\cdot\vec{i} - \cos\phi\cdot\vec{j}$

$\vec{a}\cdot\vec{n} = 0$

$|\overline{AB}| = L$

Parameter-Darstellung von $\overline{AB}$

$$\vec{r}\,' = \vec{r}_A\,' + \lambda\vec{a}$$

$$ds = d\lambda \quad , \quad 0 \le \lambda \le L$$

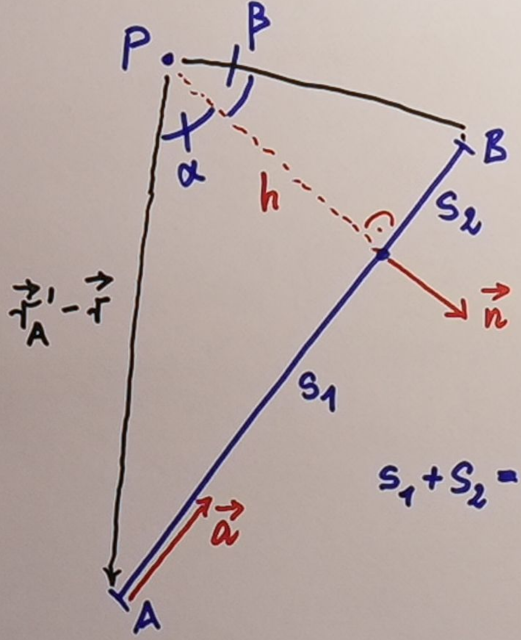
$$(\vec{r}\,'-\vec{r})\vec{n} = (\vec{r}_A\,'-\vec{r})\vec{n} = \text{const}$$

$$|\vec{r}\,'-\vec{r}|^2 = \lambda^2 + 2\left[(\vec{r}_A\,'-\vec{r})\cdot\vec{a}\right]\cdot\lambda + \ell^2 \quad , \quad \ell = |\vec{r}_A\,'-\vec{r}|$$

$$\Delta\theta = \left[(\vec{r}_A\,'-\vec{r})\vec{n}\right]\int_0^L \frac{d\lambda}{\lambda^2 + 2\left[(\vec{r}_A\,'-\vec{r})\cdot\vec{a}\right]\cdot\lambda + \ell^2}$$

$$= \arctan\frac{(\vec{r}-\vec{r}_A\,')\vec{a}}{(\vec{r}_A\,'-\vec{r})\vec{n}} + \arctan\frac{L-(\vec{r}-\vec{r}_A\,')\vec{a}}{(\vec{r}_A\,'-\vec{r})\vec{n}}$$

$$\Delta\theta = \arctan \frac{(\vec{r} - \vec{r}_A{}')\vec{a}}{(\vec{r}_A{}' - \vec{r})\vec{n}} + \arctan \frac{L - (\vec{r} - \vec{r}_A{}')\vec{a}}{(\vec{r}_A{}' - \vec{r})\vec{n}}$$



$s_1 + s_2 = L$

$(\vec{r}_{A'} - \vec{r})\vec{n} = h$

$(\vec{r} - \vec{r}_A{}')\vec{a} = s_1$

$$\Delta\theta = \arctan \frac{s_1}{h} + \arctan \frac{s_2}{h}$$

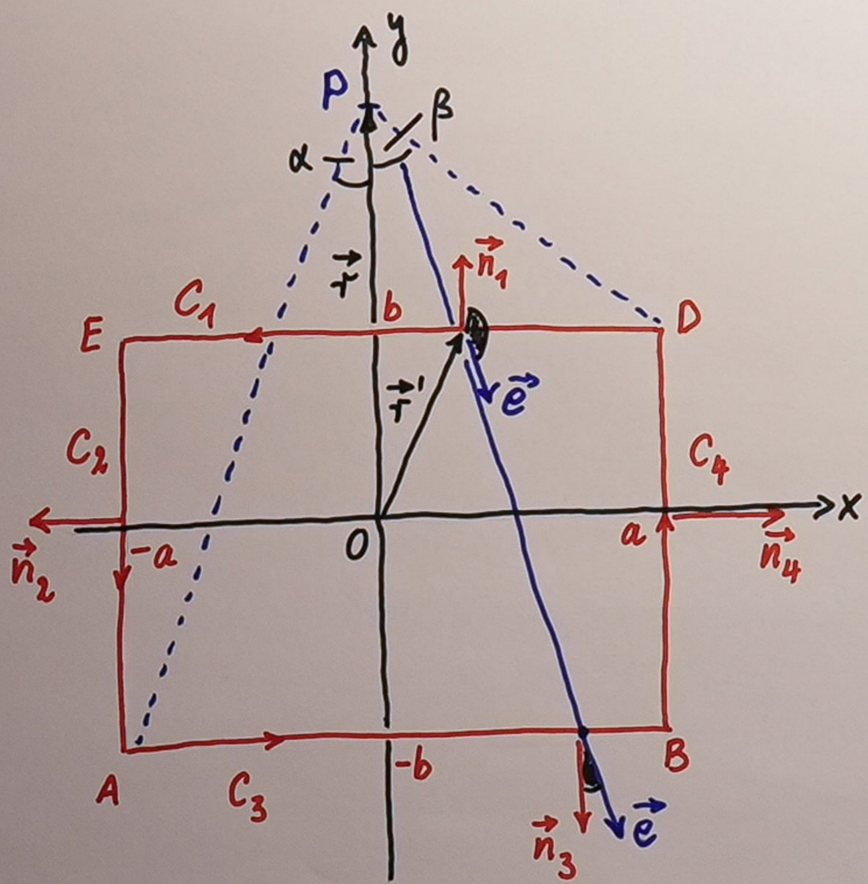
$$= \arctan(\tan\alpha) + \arctan(\tan\beta)$$

$$\Delta\theta = \alpha + \beta$$

Der ebene Winkel als Linienintegral – Beispiele

Beispiel 2: Rechteck

$$\Delta\theta = \int_C \frac{\vec{e}\cdot\vec{n}}{|\vec{r}\,' - \vec{r}|}\,ds = \int_C \frac{(\vec{r}\,'-\vec{r})\cdot\vec{n}}{|\vec{r}\,'-\vec{r}|^2}\,ds$$



C_1: $\overrightarrow{DE}$ $\quad \vec{a} = -\vec{i}\,, \quad \vec{n}_1 = \vec{j}$

$\vec{r}\,' = \vec{r}_D\,' - \lambda\vec{i}\,, \quad \vec{r}_D\,' = a\vec{i} + b\vec{j}$

$L = 2a \rightarrow 0 \leq \lambda \leq 2a$

$(\vec{r}_D\,' - \vec{r})\vec{a} = -a$

$(\vec{r}_D\,' - \vec{r})\vec{n} = -(y_P - b)$

$\rightarrow \Delta\theta_1 = -2\beta\,, \quad \tan\beta = \dfrac{a}{y_P - b}$

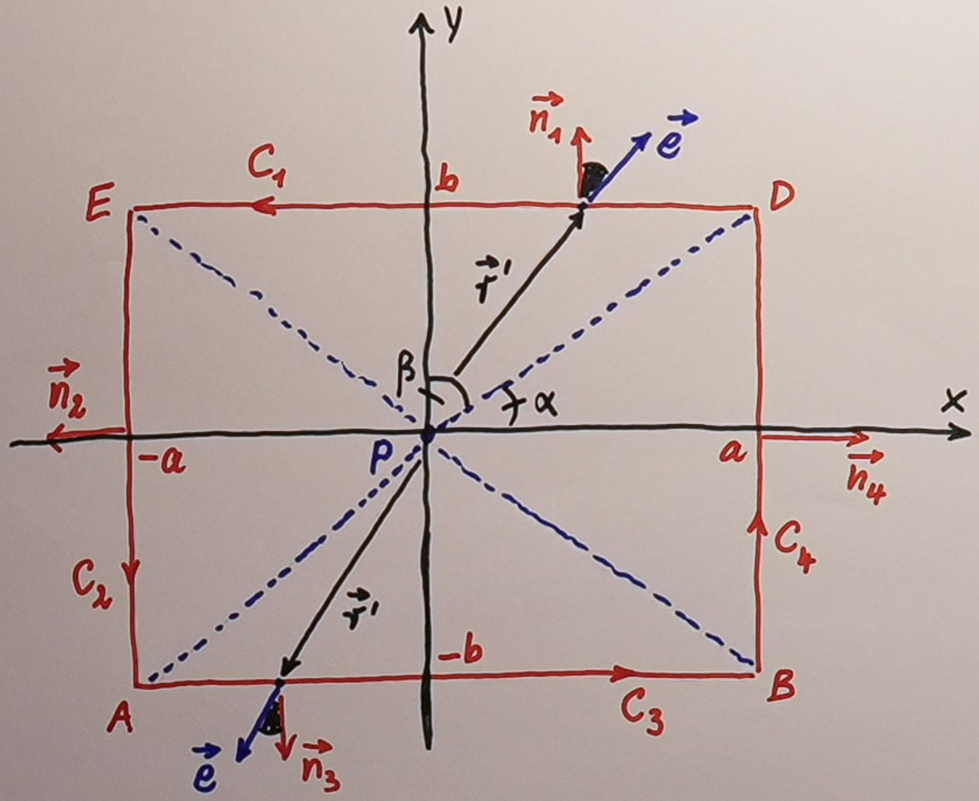
C_3: $\Delta\theta_3 = 2\alpha\,, \quad \tan\alpha = \dfrac{a}{y_P + b}$

C_2: $\Delta\theta_2 = \beta - \alpha$

C_4: $\Delta\theta_4 = \beta - \alpha$

P: $\vec{r} = y_P \cdot \vec{j}\,, \quad y_P > b$

Resultat: $\underline{\underline{\Delta\theta}} = \Delta\theta_1 + \Delta\theta_2 + \Delta\theta_3 + \Delta\theta_4 = \underline{\underline{0.}}$



P: $\vec{r} \equiv \vec{0}$

$$C_1: \quad (\vec{r}_D{}' - \vec{r})\,\vec{a} = -a$$

$$(\vec{r}_D{}' - \vec{r})\,\vec{n} = b$$

$$\longrightarrow \Delta\theta_1 = \Delta\theta_3 = 2\beta\,, \quad \tan\beta = \frac{a}{b}$$

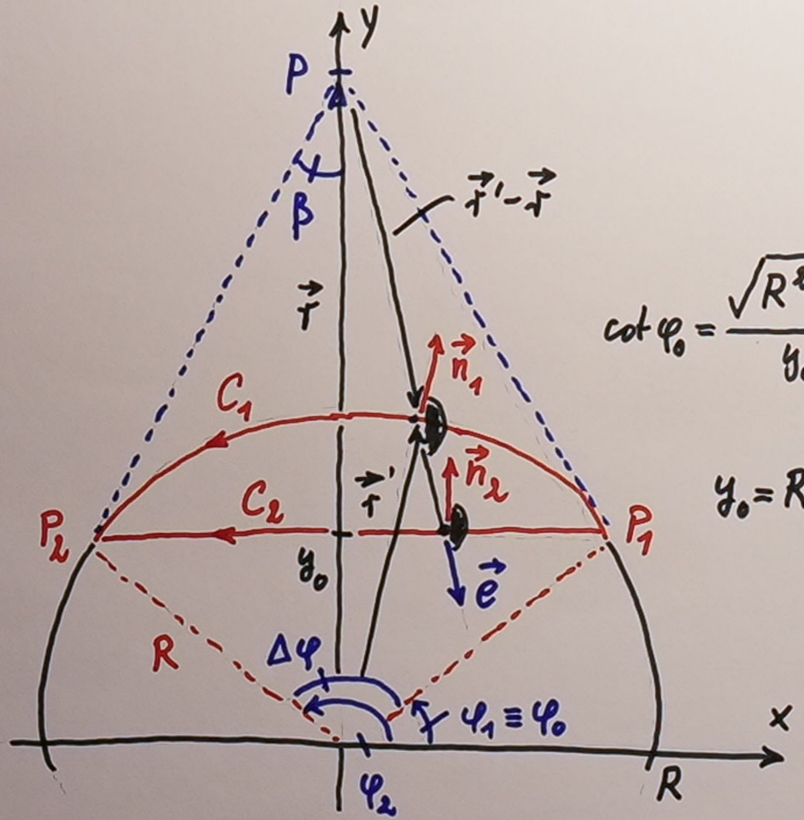
$$\Delta\theta_2 = \Delta\theta_4 = 2\alpha\,, \quad \tan\alpha = \frac{b}{a}$$

$$\alpha + \beta = \frac{\pi}{2}$$

Resultat: $\underline{\underline{\Delta\theta}} = 4(\alpha+\beta) = \underline{\underline{2\pi}}$ wie zu erwarten

(„Rundumblick")

Beispiel 3: Kreisbogen $\overset{\frown}{P_1 P_2}$



$$P: \vec{r} = y_P \vec{j}, \quad y_P > R$$

$$\vec{r}' = R \cdot \vec{n}_1 = R\left(\cos\varphi \cdot \vec{i} + \sin\varphi \, \vec{j}\right)$$

$$|\vec{r}' - \vec{r}| = (R^2 + r^2) - 2rR\sin\varphi$$

$$(\vec{r}' - \vec{r}) \cdot \vec{n}_1 = R - r\sin\varphi$$

$$\varphi_0 \le \varphi \le \pi - \varphi_0, \quad \Delta\varphi = \varphi_2 - \varphi_1 = \pi - 2\varphi_0$$

$$\Delta\theta_1 = R \int_{\varphi_0}^{\pi - \varphi_0} \frac{R - r\sin\varphi}{(R^2 + r^2) - 2rR\sin\varphi} d\varphi$$

$$= \frac{1}{2}\Delta\varphi - \arctan \frac{(R^2 + r^2)\tan\frac{\varphi}{2} - 2rR}{r^2 - R^2} \Bigg|_{\varphi = \varphi_0}^{\pi - \varphi_0}$$

$$\cot\varphi_0 = \frac{\sqrt{R^2 - y_0^2}}{y_0}, \qquad y_0 = R\sin\varphi_0$$

$$= \left(\frac{\pi}{2} - \varphi_0\right) - \arctan\left[\frac{(r^2 - R^2)\, y_0}{(r^2 + R^2)\, y_0 - 2rR^2} \cdot \cot\varphi_0\right]$$

$$\longrightarrow \quad \tan\Delta\theta_1 = 2 \cdot \frac{(y_0 - r)\sqrt{R^2 - y_0^2}}{2y_0(y_0 - r) + (r^2 - R^2)}$$

Diskussion:

- $C_2: \overrightarrow{P_1 P_2} \qquad \vec{n}_2 = \vec{j} \longrightarrow \Delta\theta_2 = -2\beta$

$$\tan\beta = \frac{\sqrt{R^2 - y_0^2}}{r - y_0}$$

Resultat: $\tan\Delta\theta_2 = \tan\Delta\theta_1$

$\rightarrow \Delta\theta_2 = \Delta\theta_1$

$\rightarrow C_2: \overrightarrow{P_2 P_1}, \quad \vec{n}_2 = -\vec{j}$

$\Delta\theta_1 + \Delta\theta_2 = 0$

– $\overline{PP_1}$ und $\overline{PP_2}$ berühren Kreis tangential

$$y_0 = \frac{R^2}{r}$$

$$\tan \Delta\theta_1 = -2 \cdot \frac{R\sqrt{r^2 - R^2}}{r^2 - 2R^2} = -\tan 2\beta$$

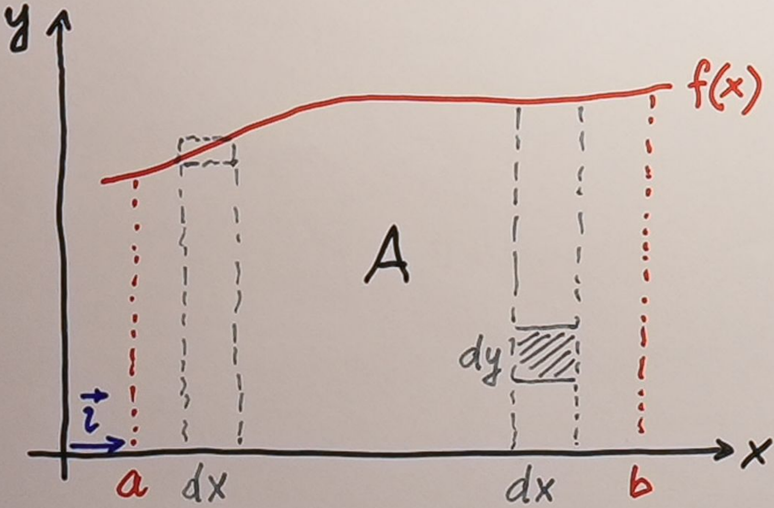
$$\underline{\underline{\Delta\theta_1 = -2\beta = -2\varphi_0}}$$

$\underline{r \gg R}$:
$$\tan \Delta\theta_1 \approx -2 \cdot \frac{R}{r} \ll 1$$

$$\Delta\theta_1 \approx -2 \cdot \frac{R}{r} \, .$$

Oberflächenintegrale

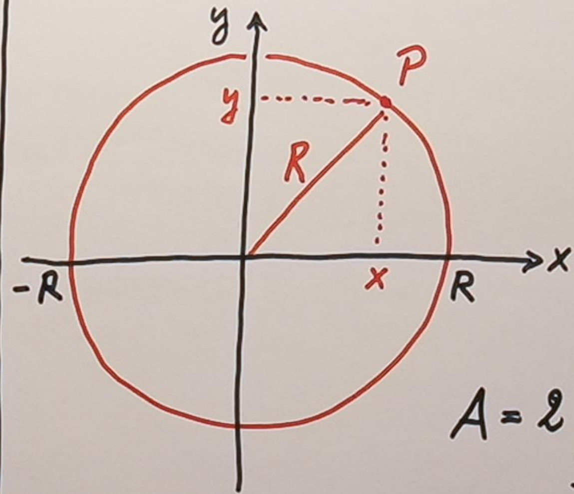
Das bestimmte Integral – oder: die ebene Fläche als Linienintegral



$$A = \int_a^b f(x)\,dx$$

Linienintegral: $C: \vec{r} = x \cdot \vec{i}$

Beispiel: von einem Kreis umschlossene Fläche



$$x^2 + y^2 = R^2, \quad R = const$$

$$y = f(x) = \sqrt{R^2 - x^2}$$

$$A = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx$$

$$A = 4\int_0^R \sqrt{R^2 - x^2}\,dx$$

$$= 4 \cdot \frac{1}{2}\left(x\sqrt{R^2 - x^2} + R^2 \arcsin\frac{x}{R}\right)\Big|_0^R$$

$$= 2R^2 \arcsin 1 = 2R^2 \cdot \frac{\pi}{2}$$

$$= \pi R^2$$

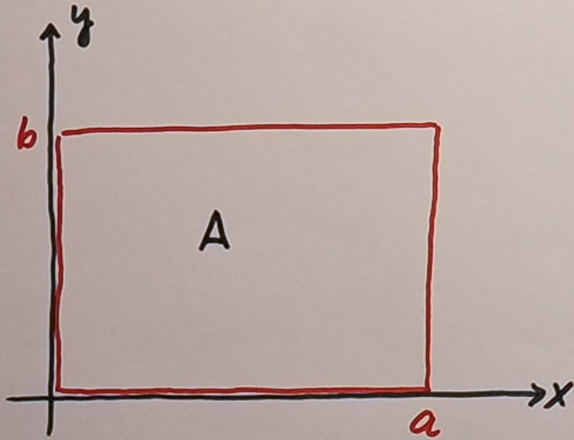
Die ebene Fläche als Doppelintegral

„Höhe" $f(x)$:

$$\int_0^{f(x)} dy = y\Big|_0^{f(x)} = f(x)$$

$$A = \int_{x=a}^{b}\left[\int_{y=0}^{f(x)} dy\right]dx = \int_{x=a}^{b}\int_{y=0}^{f(x)} dy\,dx\ , \text{ Doppel-integral}$$

Beispiel 1:



$$A = \int_{x=0}^{a}\int_{y=0}^{b} dy\,dx$$

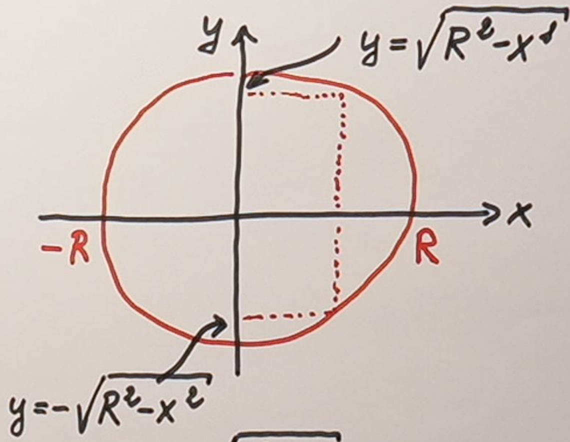
$$= \int_{x=0}^{a} b\,dx = b\cdot a$$

$$A = \left[\int_{x=0}^{a} dx\right]\cdot\left[\int_{y=0}^{b} dy\right]$$

$$= a\cdot b$$

$$A = \int_{y=0}^{b}\int_{x=0}^{a} dx\,dy = \int_{y=0}^{b} a\,dy = a\cdot b$$

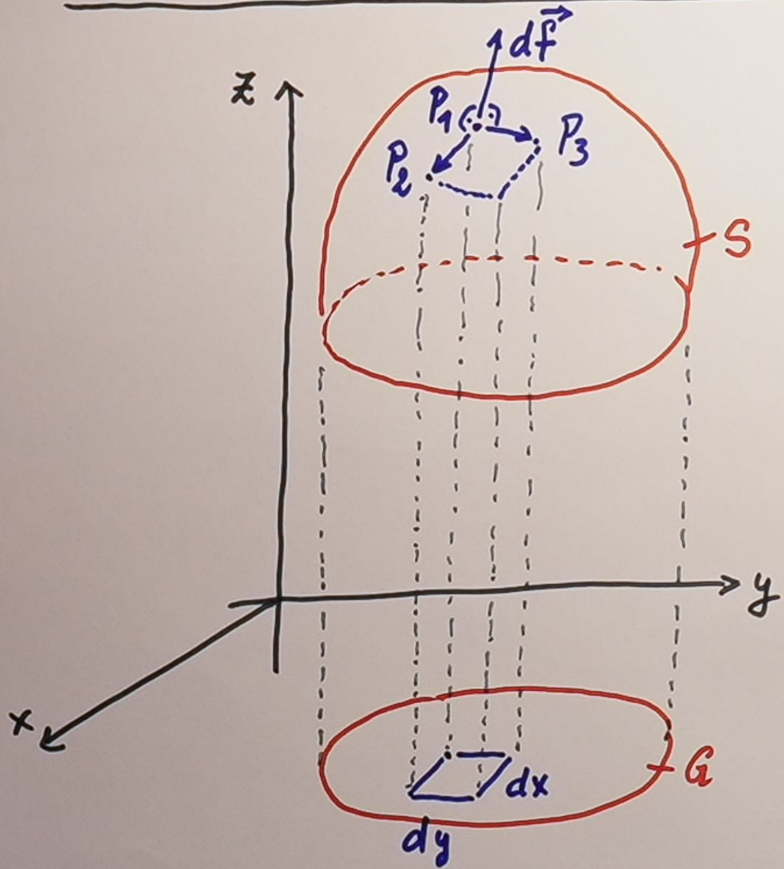
Beispiel 2:



$$A = \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy\,dx$$

$$= 2\int_{x=-R}^{R}\sqrt{R^2-x^2}\,dx$$

Oberflächenintegrale gekrümmter Flächen



$$S: z = z(x,y)$$

$$\overrightarrow{P_1P_2} = dx \cdot \vec{i} + \frac{\partial z}{\partial x} dx \cdot \vec{k}$$

$$\overrightarrow{P_1P_3} = dy \cdot \vec{j} + \frac{\partial z}{\partial y} dy \cdot \vec{k}$$

$$d\vec{f} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & 0 & \frac{\partial z}{\partial x} dx \\ 0 & dy & \frac{\partial z}{\partial y} dy \end{vmatrix} = \left(-\frac{\partial z}{\partial x} \vec{i} - \frac{\partial z}{\partial y} \vec{j} + \vec{k} \right) dx\, dy$$

$$= \vec{n} \cdot df \longrightarrow \vec{n} = \frac{-\frac{\partial z}{\partial x} \vec{i} - \frac{\partial z}{\partial y} \vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

$$P_1: \quad x,\ y,\ z(x,y)$$

$$P_2: \quad x+dx,\ y,\ z(x+dx,y) = z(x,y) + \frac{\partial z}{\partial x} dx$$

$$P_3: \quad x,\ y+dy,\ z(x,y+dy) = z(x,y) + \frac{\partial z}{\partial y} dy$$

$$\boxed{df = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dx\, dy}$$

$$A_S = \iint_S df = \iint_G \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\, dy$$

<u>Beispiel</u>: Flächeninhalt der Oberfläche einer Halbkugel

$S: \; x^2 + y^2 + z^2 = R^2 \;, \quad z > 0$

$G: \; x^2 + y^2 \leq R^2$

Flächenelement: $df = \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}\, dx\, dy = \frac{R}{z}\, dx\, dy$

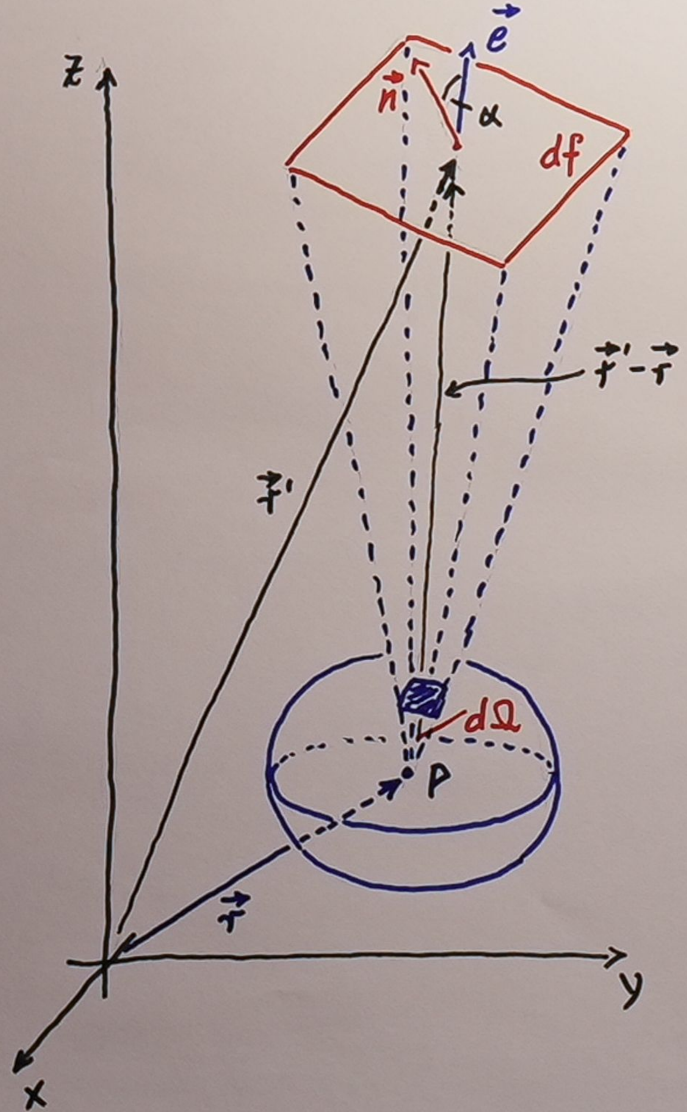
$$A_S = R \iint_G \frac{dx\, dy}{\sqrt{R^2 - x^2 - y^2}} = R \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{dy}{\sqrt{R^2 - x^2 - y^2}}$$

$$= 2R \int_{-R}^{R} \arcsin \frac{y}{\sqrt{R^2 - x^2}} \Bigg|_{y=0}^{\sqrt{R^2-x^2}} dx$$

$$= 2R \int_{-R}^{R} \arcsin 1\, dx$$

$$= 2\pi R^2$$

Der Raumwinkel als Oberflächenintegral



ebener Winkel, Linienintegral:

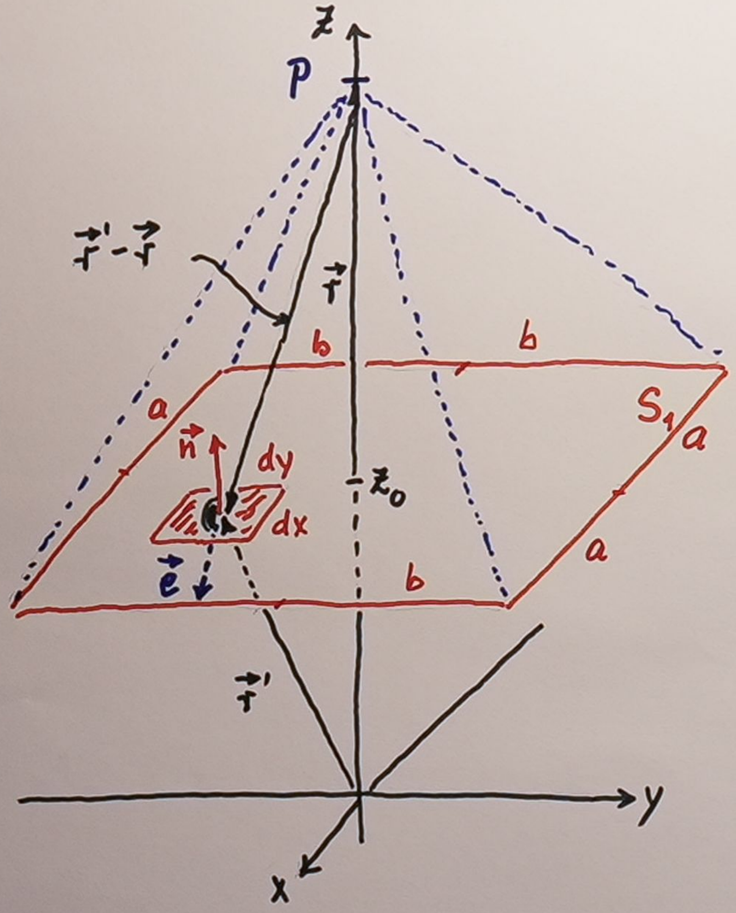
$$\Delta\theta = \int_C \frac{\vec{e}\cdot\vec{n}}{|\vec{r}\,'-\vec{r}|}\,ds = \int_C \frac{(\vec{r}\,'-\vec{r})\cdot\vec{n}}{|\vec{r}\,'-\vec{r}|^2}\,ds \quad ; \quad \vec{e} = \frac{\vec{r}\,'-\vec{r}}{|\vec{r}\,'-\vec{r}|}$$

Verallgemeinerung:

$$\Delta\Omega = \iint_S \frac{\vec{e}\cdot\vec{n}}{|\vec{r}\,'-\vec{r}|^2}\,df = \iint_S \frac{(\vec{r}\,'-\vec{r})\,d\vec{f}}{|\vec{r}\,'-\vec{r}|^3} \quad , \quad d\vec{f} = \vec{n}\cdot df$$

Beispiel 1 : Rechteck

$$\Delta\Omega = \iint\limits_S \frac{\vec{e}\cdot\vec{n}}{|\vec{r}\,'-\vec{r}|^2}\,df = \iint\limits_S \frac{(\vec{r}\,'-\vec{r})\cdot d\vec{f}}{|\vec{r}\,'-\vec{r}|^3}$$



$$\Delta\Omega_1 = \int_{-a}^{a} dx \int_{-b}^{b} dy \; \frac{z_0 - r}{\left[x^2+y^2+(z_0-r)^2\right]^{3/2}}$$

$$= 4\cdot \arctan \frac{ab}{(z_0-r)\sqrt{a^2+b^2+(z_0-r)^2}}$$

P: $\vec{r} = r\vec{k}$

$-a \le x \le a$	$d\vec{f} = dx\,dy\cdot\vec{k}$
$-b \le y \le b$	$\vec{r}\,' = x\vec{i} + y\vec{j} + z_0\vec{k}$
$\vec{n} = \vec{k}$	$\longrightarrow (\vec{r}\,'-\vec{r})\cdot\vec{n} = z_0 - r$

Diskussion

– $r \gg a, b, z_0$

$$\Delta\Omega_1 \approx -\frac{2a\cdot 2b}{r^2} = -\frac{A_R}{r^2}.$$

– zweites Rechteck : $z_0 \longrightarrow -z_0$, $d\vec{f} \longrightarrow -d\vec{f}$; P: $\vec{r} \equiv \vec{0}$

$$\Delta\Omega_2 = 4\cdot\arctan\frac{ab}{(z_0+r)\sqrt{a^2+b^2+(z_0+r)^2}}$$

Würfel: $z_0 = b = a$

$$\longrightarrow \Delta\Omega_1 + \Delta\Omega_2 = 2\cdot 4\arctan\left(\tfrac{1}{3}\sqrt{3}\right) = 8\cdot\frac{\pi}{6} = \frac{4}{3}\pi$$

Resultat: $\underline{\underline{\Omega_{tot}}} = 3\cdot\frac{4}{3}\pi = \underline{\underline{4\pi}}$

Beispiel 2: Kreisscheibe

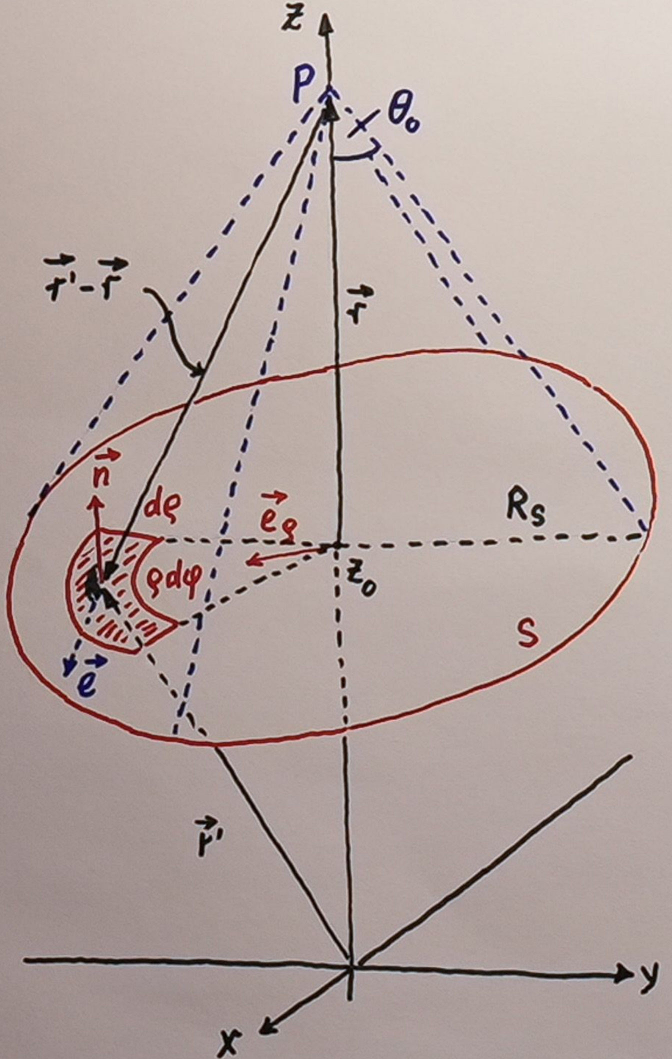
$P: \ \vec{r} = r\vec{k}$

$d\vec{f} = \varrho\, d\varrho\, d\varphi \cdot \vec{k}$

$\vec{n} = \vec{k}$

$\vec{r}\,' = \varrho\, \vec{e}_\varrho + z_0 \vec{k}$

$(\vec{r}\,' - \vec{r})\,\vec{n} = z_0 - r$



$$\Delta\Omega = \int_0^{2\pi} d\varphi \int_0^{R_S} d\varrho \cdot \varrho \, \frac{z_0 - r}{\left[\varrho^2 + (z_0 - r)^2\right]^{3/2}}$$

$$= 2\pi \cdot \delta \cdot \left(1 - \frac{1}{\sqrt{1 + \frac{R_S^2}{(r - z_0)^2}}}\right)$$

mit

$$\delta \equiv \frac{z_0 - r}{|z_0 - r|} = \begin{cases} -1 & \text{für } r > z_0 \\ +1 & \text{für } r < z_0 \end{cases}$$

Diskussion: $-\ r \gg R_S, z_0$

$$\Delta\Omega \approx -2\pi\left[1 - \left(1 - \frac{1}{2}\frac{R_S^2}{r^2}\right)\right] = -\frac{\pi R_S^2}{r^2} = -\frac{A_S}{r^2}$$

$-$ sei $r > z_0$

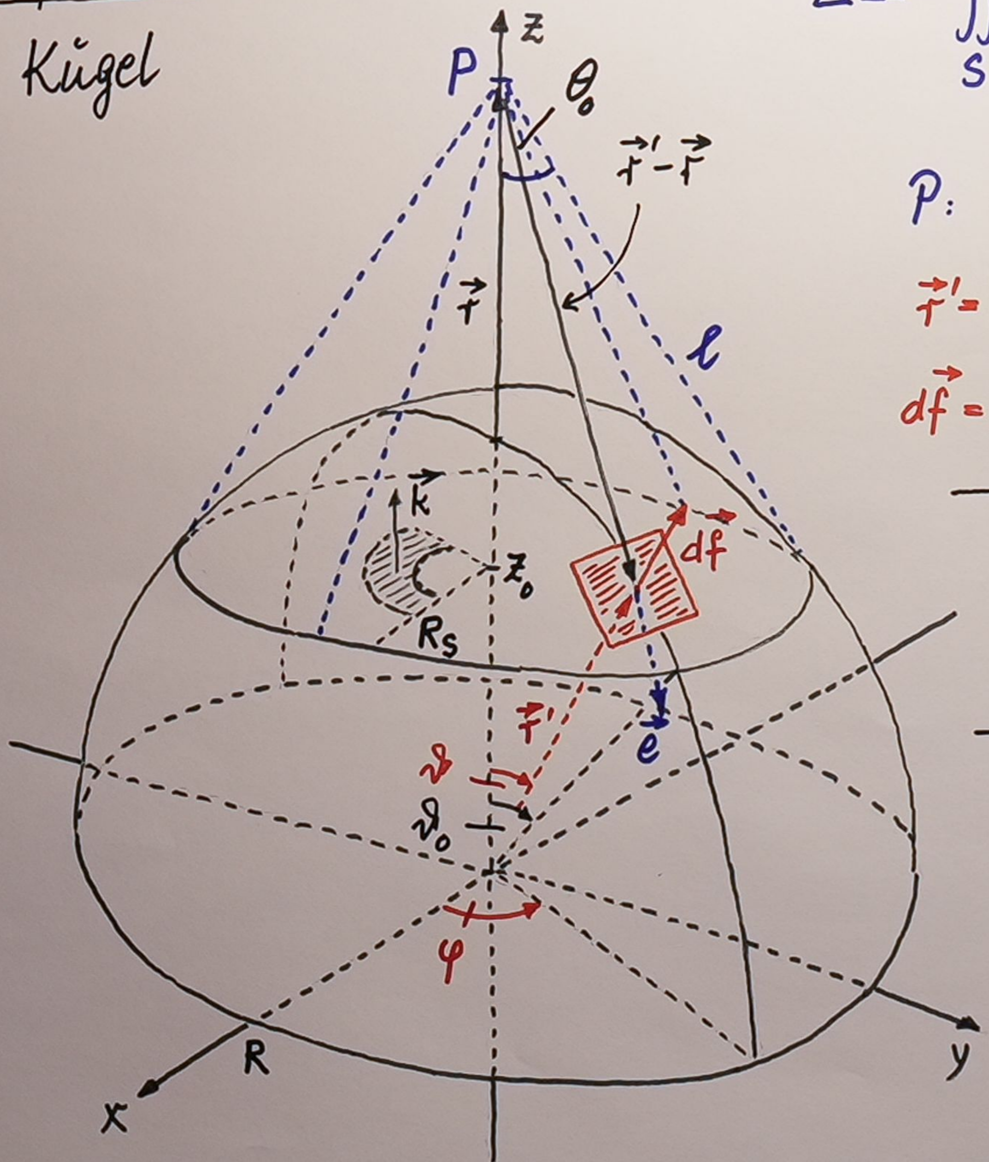
$$\tan\theta_0 = \frac{R_S}{r - z_0}$$

$$\Delta\Omega = -2\pi\left(1 - \cos\theta_0\right)$$

Der Raumwinkel als Oberflächenintegral – Beispiele

Beispiel 3:

Kugel



$$\Delta\Omega = \iint_S \frac{\vec{e}\cdot\vec{n}}{|\vec{r}'-\vec{r}|^2}\,df = \iint_S \frac{(\vec{r}'-\vec{r})\cdot d\vec{f}}{|\vec{r}'-\vec{r}|^3}$$

$$P: \quad \vec{r} = r\vec{k} = z_P \cdot \vec{k}$$

$$\vec{r}' = R\cdot\vec{e}_r \quad , \quad \vec{n} = \vec{e}_r$$

$$d\vec{f} = R^2 \sin\vartheta \, d\vartheta \, d\varphi \cdot \vec{e}_r$$

$$(\vec{r}'-\vec{r})\vec{n} = R - r\cos\vartheta$$

$$|\vec{r}'-\vec{r}|^2 = (R^2 + r^2) - 2rR\cos\vartheta$$

$$0 \le \varphi < 2\pi$$

$$0 \le \vartheta_0 \le \pi$$

Raumwinkel:
$$\Delta\Omega = R^2 \int_0^{2\pi} d\varphi \int_0^{\vartheta_0} d\vartheta \, \frac{(R - r\cos\vartheta)\cdot \sin\vartheta}{\left[(R^2+r^2) - 2rR\cos\vartheta\right]^{3/2}}$$

$$= 2\pi \left(\frac{r - R\cos\vartheta_0}{\sqrt{(R^2+r^2) - 2rR\cos\vartheta_0}} - \delta \right) , \quad \delta = \begin{cases} +1 & \text{für } r>R \\ -1 & \text{für } r<R \end{cases}$$

Diskussion

a) Vollkugel: $\vartheta_0 = \pi$

$$\Delta\Omega = 2\pi(1-\delta) = \begin{cases} 0 & \text{für } r>R \\ 4\pi & \text{für } r<R \end{cases}$$

b) Beobachter im Kugelmittelpunkt: $r=0$, $\vec{n} = \vec{e}_r = e$

$$\Delta\Omega = 2\pi(1-\cos\vartheta_0)$$

$$A = R^2 \int_0^{2\pi} d\varphi \int_0^{\vartheta_0} d\vartheta \, \sin\vartheta = 2\pi R^2 (1-\cos\vartheta_0) = R^2 \cdot \Delta\Omega$$

c) cos-Satz: $R^2 + r^2 - 2rR\cos\vartheta_0 = \ell^2$

$$\ell^2 = (r - z_0)^2 + R_S^2$$

$r > R$: $\delta = +1$

$$\Delta\Omega = -2\pi\left(1 - \frac{1}{\sqrt{1 + \frac{R_S^2}{(r-z_0)^2}}}\right)$$, vergleiche Kreisscheibe

d) Kreiskegel, halber Öffnungswinkel θ_0

$$\left.\begin{array}{l} \Delta\Omega = -2\pi\left(1 - \cos\theta_0\right) \\ \text{Kugel mit P als Mittelpunkt: } A_{(P)} = 2\pi R_{(P)}^2\left(1 - \cos\theta_0\right) \end{array}\right\} \Delta\Omega = -\frac{A_{(P)}}{R_{(P)}^2}$$

e) $r > R$, „Sehstrahlen" berühren Kugel tangential

$$\vartheta_0 = \frac{\pi}{2} - \theta_0\,, \quad \cos\theta_0 = \frac{R}{r}$$

$$\Delta\Omega = -2\pi\left(1 - \sqrt{1 - \frac{R^2}{r^2}}\right) = -2\pi\left(1 - \sin\vartheta_0\right)$$

speziell: $r \gg R$

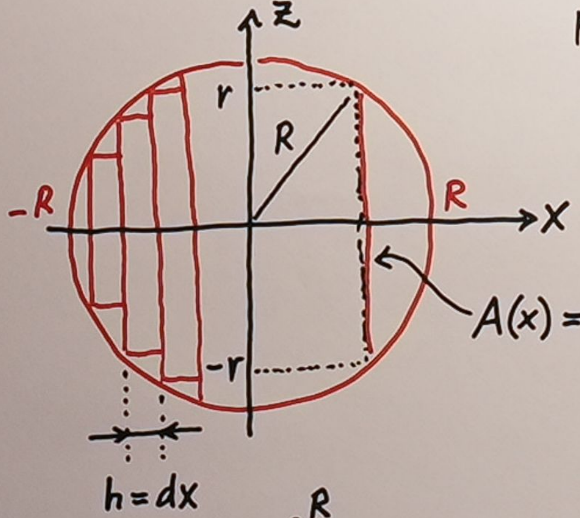
$$\Delta\Omega \approx -\frac{\pi R^2}{r^2}$$, πR^2: Äquatorfläche

Volumenintegrale

Das Volumen als Doppelintegral

Beispiel: Kugel, Radius $R = \text{const}$

$$r^2 = R^2 - x^2$$



$$A(x) = \pi\,[r(x)]^2$$

Volumen: $$V = \int_{-R}^{R} A(x)\cdot dx$$

$$= \pi \int_{-R}^{R} (R^2 - x^2)\,dx$$

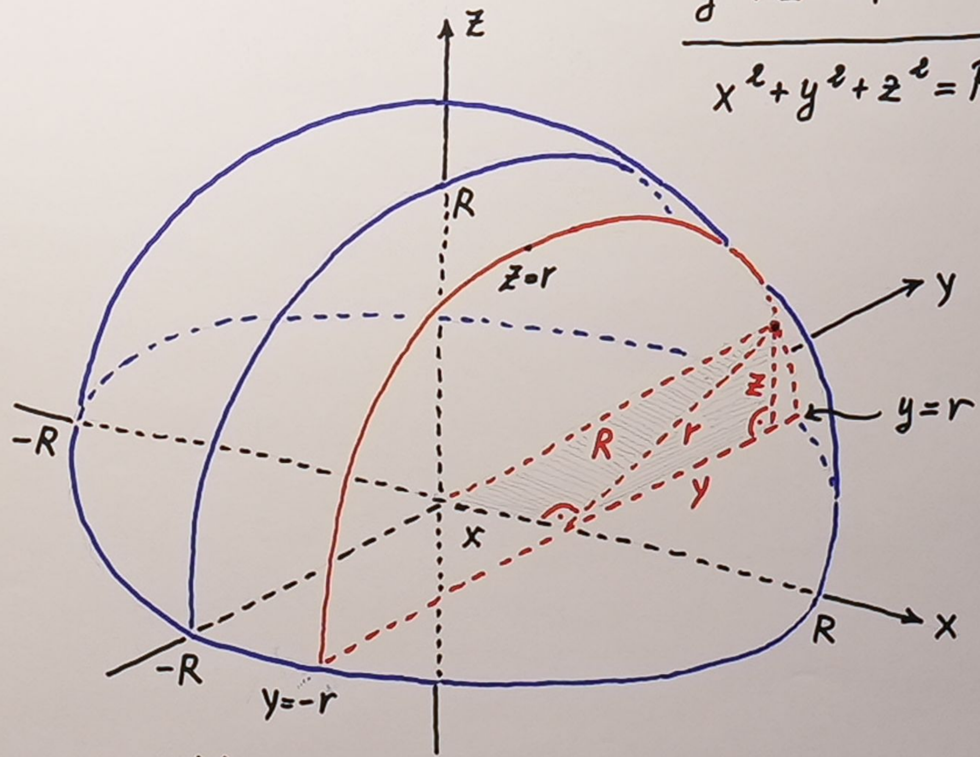
$$= 2\pi \int_{0}^{R} (R^2 - x^2)\,dx$$

$$= 2\pi \left(R^2 x - \frac{x^3}{3}\right)\Bigg|_0^R = \frac{4\pi}{3} R^3$$

A(x) als Integral

$$x^2 + r^2 = R^2$$
$$y^2 + z^2 = r^2$$

$$x^2 + y^2 + z^2 = R^2$$



$$A(x) = 2\int_{-r(x)}^{r(x)} z(x,y)\,dy$$

Volumen: $$V = \int_{x=-R}^{R} \left[2 \int_{y=-r(x)}^{r(x)} z(x,y)\,dy \right] dx$$

$$= 2 \int_{x=-R}^{R} \int_{y=-r(x)}^{r(x)} z(x,y)\,dy\,dx$$

$$V = 2\int_{x=-R}^{R}\int_{y=-r(x)}^{r(x)} z(x,y)\,dx\,dy$$

$$r(x) = \sqrt{R^2-x^2}$$

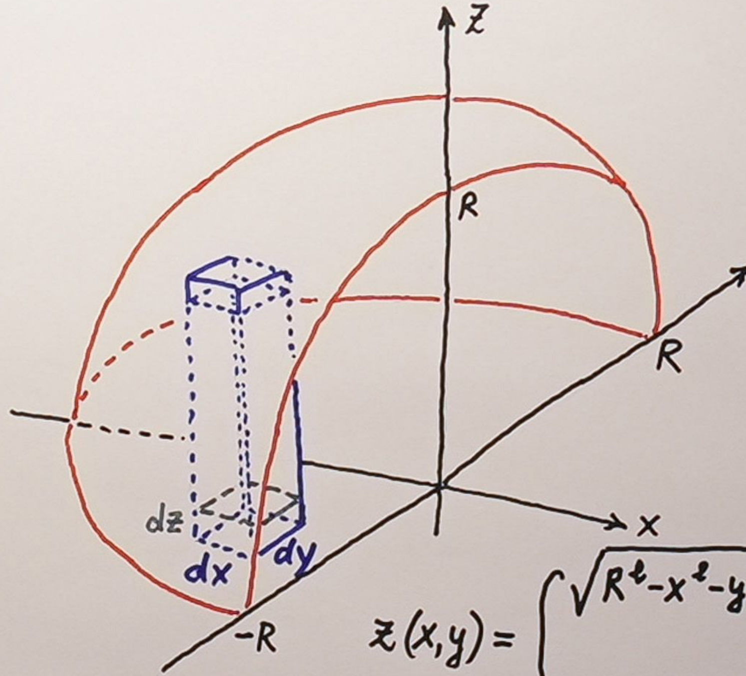
$$z = \sqrt{R^2-x^2-y^2}$$

$$= 2\int_{x=-R}^{R} 2\cdot\frac{1}{2}\left[y\sqrt{R^2-x^2-y^2} + (R^2-x^2)\arcsin\frac{y}{\sqrt{R^2-x^2}}\right]_{y=0}^{\sqrt{R^2-x^2}} dx$$

$$= 4\int_{x=0}^{R}(R^2-x^2)\cdot\frac{\pi}{2}\,dx$$

$$= 2\pi\left(R^2x-\frac{1}{3}x^3\right)\Big|_0^R = \frac{4}{3}\pi R^3$$

Dreifach integrale



$$z(x,y) = \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz$$

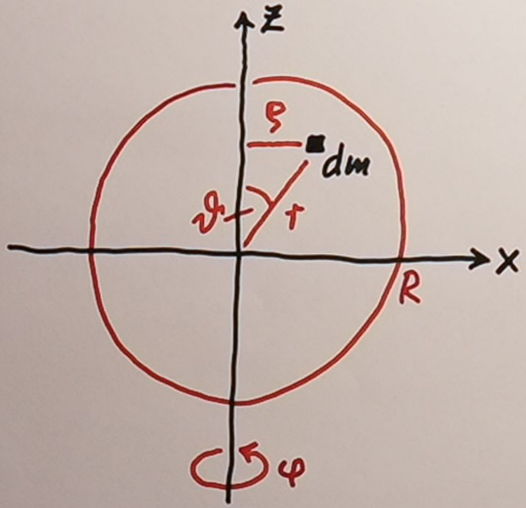
Volumen:

$$V = \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{z=-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz\,dy\,dx$$

$$= \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2-y^2}\,dy\,dx$$

Volumenintegrale : Beispiele

1. Trägheitsmoment einer Kugel



Trägheitsmoment:

$$I = \int_V \varrho^2 \, dm$$

Massenelement: $dm = \mu \cdot dV$

Massendichte: $\mu = const$

$M = \mu \cdot V$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad I = \frac{M}{V} \int_{Kugel} \varrho^2 \, dV$$

Kugelkoordinaten: $dV = r^2 dr \cdot \sin\vartheta \, d\vartheta \cdot d\varphi$

$\varrho = r \sin\vartheta$

$$I = \frac{M}{V} \int_0^R r^4 dr \cdot \int_0^\pi \sin^3\vartheta \, d\vartheta \cdot \int_0^{2\pi} d\varphi$$

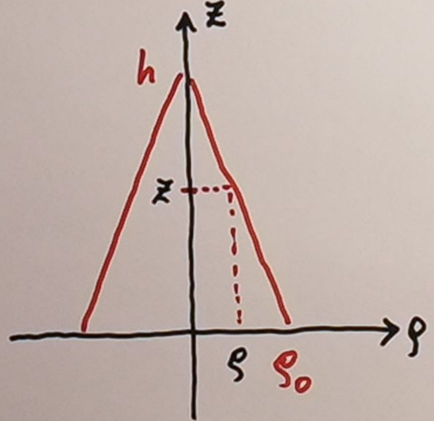
$$= \frac{M}{V} \cdot \frac{R^5}{5} \left(-\cos\vartheta + \frac{1}{3}\cos^3\vartheta \right)\Big|_0^\pi \cdot 2\pi$$

$$= \frac{M}{\frac{4}{3}\pi R^3} \cdot \frac{R^5}{5} \cdot \frac{4}{3} \cdot 2\pi$$

$$\underline{\underline{I = \frac{2}{5} M R^2}} \quad , \quad \frac{2}{5} : \text{„Geometriefaktor"}$$

2. Volumen eines Kreiskegels

a) Zylinderkoordinaten



$$\frac{h}{\varrho_0} = \frac{z}{\varrho_0 - \varrho}$$

$$\rightarrow \quad z = \frac{h}{\varrho_0}(\varrho_0 - \varrho)$$

Zylinderkoordinaten:

$$dV = \varrho \cdot d\varrho \cdot dz \cdot d\varphi$$

$$V = \int_{\varrho=0}^{\varrho_0} d\varrho \cdot \varrho \int_{z=0}^{\frac{h}{\varrho_0}(\varrho_0-\varrho)} dz \cdot \int_0^{2\pi} d\varphi$$

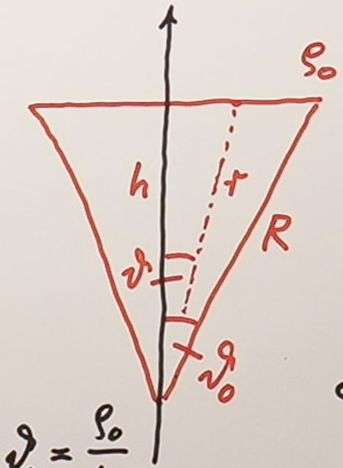
$$= 2\pi \int_{\varrho=0}^{\varrho_0} \varrho \; z \Big|_0^{\frac{h}{\varrho_0}(\varrho_0-\varrho)} d\varrho$$

$$= 2\pi \frac{h}{\varrho_0} \int_{\varrho=0}^{\varrho_0} \varrho(\varrho_0 - \varrho)\, d\varrho$$

$$= 2\pi \frac{h}{\varrho_0}\left(\varrho_0 \frac{\varrho^2}{2} - \frac{1}{3}\varrho^3\right)\Big|_0^{\varrho_0}$$

$$\underline{\underline{V = \frac{\pi}{3}\varrho_0^2 \cdot h}}$$

b) Kugelkoordinaten



$$\cos\vartheta = \frac{h}{r}$$

$$r = \frac{h}{\cos\vartheta}, \quad 0 \leq \vartheta \leq \vartheta_0$$

Volumenelement:

$$dV = r^2 dr \cdot \sin\vartheta\, d\vartheta \cdot d\varphi$$

$$\tan\vartheta_0 = \frac{\varrho_0}{h}$$

$$V = \int_{\vartheta=0}^{\vartheta_0} d\vartheta \sin\vartheta \int_{r=0}^{h/\cos\vartheta} r^2 dr \cdot \int_{\varphi=0}^{2\pi} d\varphi$$

$$= 2\pi \frac{h^3}{3} \int_{\vartheta=0}^{\vartheta_0} \frac{\sin\vartheta}{\cos^3\vartheta}\, d\vartheta = \frac{2\pi}{3} h^3 \cdot \frac{1}{2}\frac{1}{\cos^2\vartheta}\Big|_{\vartheta=0}^{\vartheta_0}$$

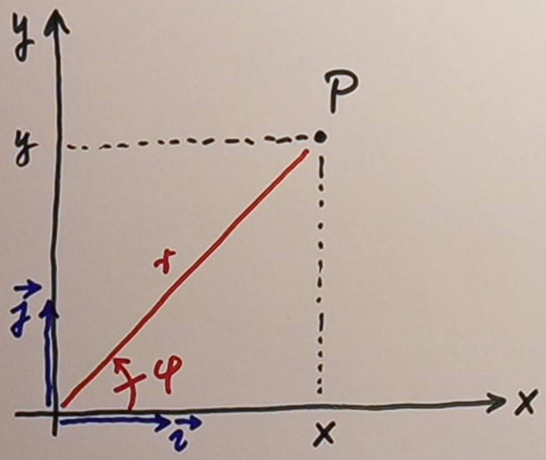
$$= \frac{\pi}{3} h^3 \left(\frac{1}{\cos^2\vartheta_0} - 1\right) = \frac{\pi}{3} h^3 \tan^2\vartheta_0$$

$$\underline{\underline{V}} = \frac{\pi}{3} h^3 \cdot \frac{\varrho_0^2}{h^2} = \underline{\underline{\frac{\pi}{3}\varrho_0^2 h}}$$

Intermezzo: Krummlinige Orthogonalkoordinaten

(Polar-, Zylinder-, Kugel-)

I. Ebene Polarkoordinaten (r, φ)

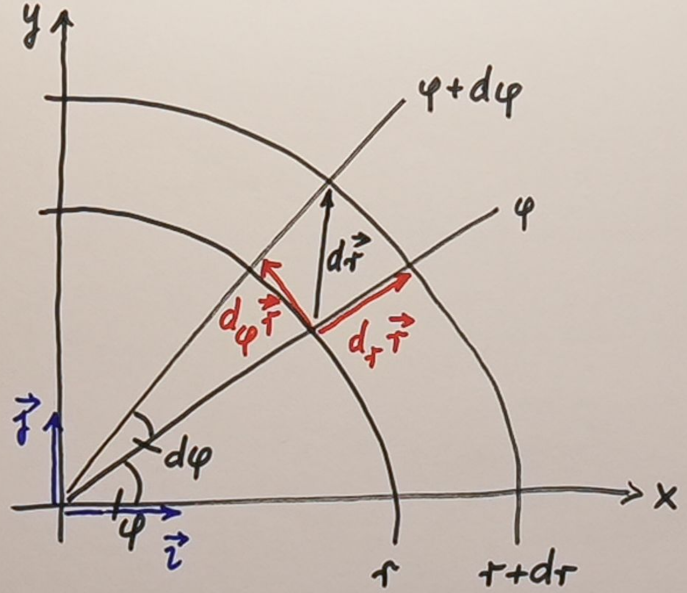


φ : Azimut

$$x = r\cos\varphi \qquad r \geq 0$$
$$y = r\sin\varphi \qquad 0 \leq \varphi < 2\pi$$

$$r = \sqrt{x^2 + y^2}$$
$$\tan\varphi = \frac{y}{x}$$

Tangentenvektoren an Koordinatenlinien



$r = \text{const}: \quad x^2 + y^2 = r^2$

$\varphi = \text{const}: \quad y = \tan\varphi \cdot x$

radial: $d\varphi = 0$

$$d_r\vec{r} = \frac{\partial \vec{r}}{\partial r} dr$$
$$= \left(\frac{\partial x}{\partial r}\vec{i} + \frac{\partial y}{\partial r}\vec{j}\right) dr$$
$$= (\cos\varphi \cdot \vec{i} + \sin\varphi\, \vec{j})\, dr$$
$$= \vec{e}_r \cdot dr$$

Abstand infinitesimal benachbarter Punkte:

$$d\vec{r} = dx \cdot \vec{i} + dy \cdot \vec{j}$$
$$= \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \varphi} d\varphi\right)\vec{i}$$
$$+ \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \varphi} d\varphi\right)\vec{j}$$
$$= \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \varphi} d\varphi$$

azimutal: $dr = 0$

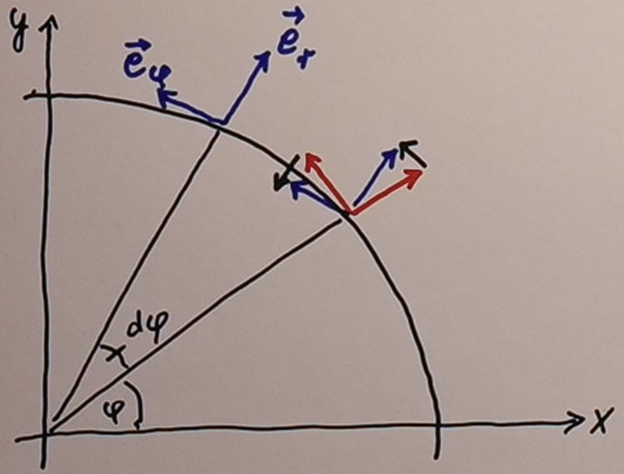
$$d_\varphi \vec{r} = \frac{\partial \vec{r}}{\partial \varphi} d\varphi = \left(\frac{\partial x}{\partial \varphi} \vec{i} + \frac{\partial y}{\partial \varphi} \vec{j} \right) d\varphi$$

$$= r\left(-\sin\varphi \cdot \vec{i} + \cos\varphi\, \vec{j}\right) d\varphi$$

$$= \vec{e}_\varphi \cdot r\, d\varphi$$

Tangenten-Einheitsvektoren

$$\left.\begin{aligned} \vec{e}_r &= \cos\varphi \cdot \vec{i} + \sin\varphi \cdot \vec{j} \\ \vec{e}_\varphi &= -\sin\varphi \cdot \vec{i} + \cos\varphi \cdot \vec{j} \end{aligned}\right\} \begin{aligned} &\vec{e}_r \cdot \vec{e}_\varphi = 0 \\ &\text{(orthogonal)} \end{aligned}$$



$$\frac{d}{d\varphi} \vec{e}_r = -\sin\varphi \cdot \vec{i} + \cos\varphi \cdot \vec{j} = \vec{e}_\varphi$$

$$\frac{d}{d\varphi} \vec{e}_\varphi = -\cos\varphi \cdot \vec{i} - \sin\varphi \cdot \vec{j} = -\vec{e}_r$$

Linienelement

Ortsvektor $\vec{r} = r \cdot \vec{e}_r(\varphi)$

$$d\vec{r} = d_r \vec{r} + d_\varphi \vec{r} = \vec{e}_r\, dr + \vec{e}_\varphi \cdot r\, d\varphi$$

$$ds^2 = d\vec{r} \cdot d\vec{r} = dr^2 + r^2 d\varphi^2$$

Beispiel: Bogenlänge einer Kurve

$$s(P_1, P_2) = \int_C ds = \int_C \sqrt{dx^2 + dy^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \quad \text{, kartesisch}$$

$$s(P_1, P_2) = \int_C \sqrt{dr^2 + r^2 d\varphi^2}$$

$$= \int_{\varphi_1}^{\varphi_2} \sqrt{\left(\frac{dr}{d\varphi}\right)^2 + r^2}\, d\varphi \quad \text{, polar}$$

$r(\varphi)$: Polargleichung

Anwendung: Kreis

Polargleichung: $r = R = \text{const}$

$$s(\varphi_1, \varphi_2) = R\int_{\varphi_1}^{\varphi_2} d\varphi = R(\varphi_2 - \varphi_1)$$

$(\varphi_2 - \varphi_1 = 2\pi \longrightarrow$ Umfang $2\pi R)$

Kartesisch: $x^2 + y^2 = R^2$

$$y' = -\frac{x}{y}$$

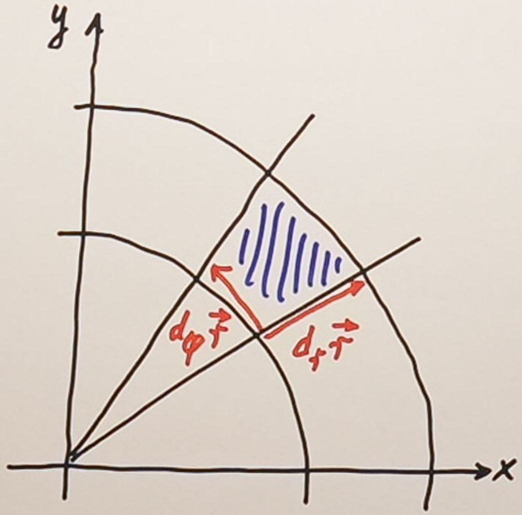
$$s(x_1, x_2) = \int_{x_1}^{x_2} \sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= \int_{x_1}^{x_2} \frac{dx}{\sqrt{1 - \left(\frac{x}{R}\right)^2}}$$

trigonometrische Substitution:

$$\frac{x}{R} = \cos\varphi$$

Flächenelement $d\vec{f}$



$$d\vec{f} = d_r\vec{r} \times d_\varphi \vec{r}$$

$$= (\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}) \times (-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j})\, dr \cdot r\,d\varphi$$

$$d\vec{f} = \vec{k} \cdot r\,dr\,d\varphi$$

Beispiel: Kreisfläche

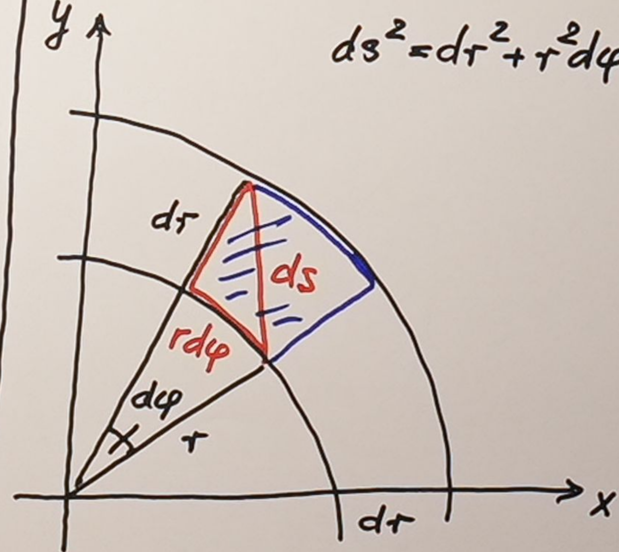
$$A = \int_{r=0}^{R} \int_{\varphi=0}^{2\pi} r\,dr\,d\varphi$$

$$= \left[\int_0^R r\,dr\right] \cdot \left[\int_0^{2\pi} d\varphi\right]$$

$$= \pi R^2$$

Anmerkung:

$$ds^2 = dr^2 + r^2 d\varphi^2$$



$$df = \frac{d\varphi}{2\pi} \cdot \pi\left[(r + dr)^2 - r^2\right]$$

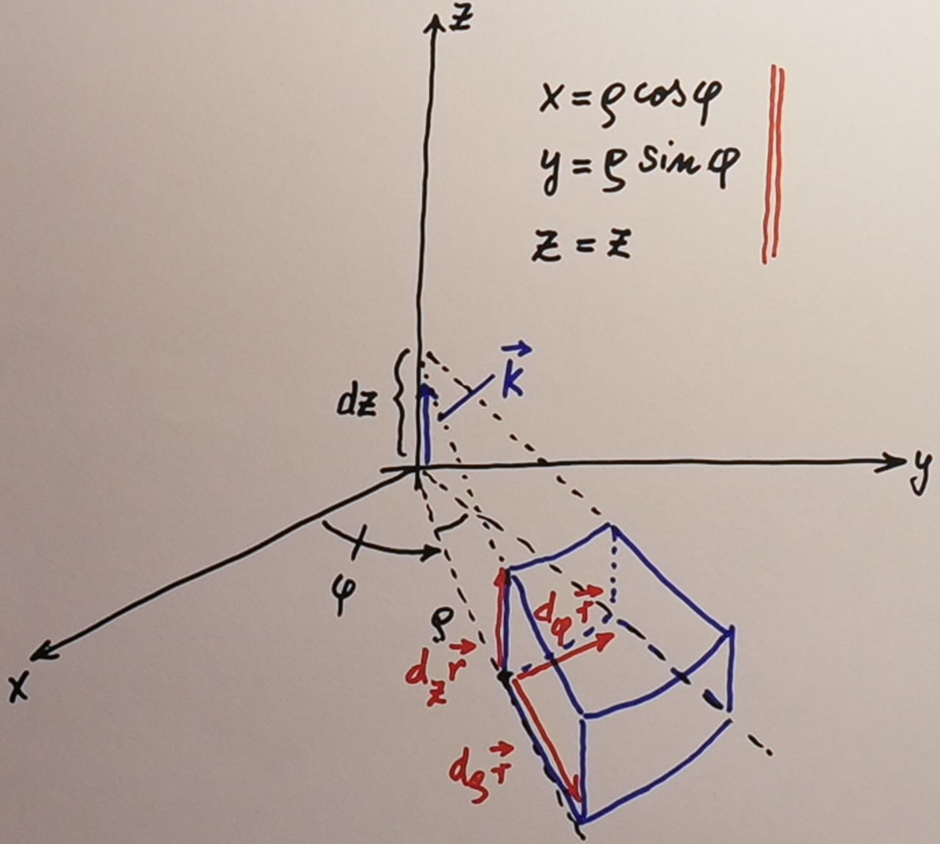
$$= r\,dr\,d\varphi\left(1 + \frac{1}{2}\frac{dr}{r}\right)$$

$$\longrightarrow r\,dr\,d\varphi$$

Intermezzo: Krummlinige Orthogonalkoordinaten

II. Zylinderkoordinaten (ϱ, φ, z)

$$x = \varrho \cos\varphi$$
$$y = \varrho \sin\varphi$$
$$z = z$$



- Koordinatenlinien:

$z = \text{const},\ \varphi = \text{const}: \quad y = \tan\varphi \cdot x$

$z = \text{const},\ \varrho = \text{const}: \quad x^2 + y^2 = \varrho^2$

$\varrho = \text{const},\ \varphi = \text{const}:$ Parallelen zur z-Achse

- Tangentenvektoren der Koordinatenlinien:

$z = \text{const},\ \varphi = \text{const}: \quad d_\varrho \vec{r} = \vec{e}_\varrho \cdot d\varrho$

$z = \text{const},\ \varrho = \text{const}: \quad d_\varphi \vec{r} = \vec{e}_\varphi \cdot \varrho\, d\varphi$

$\varrho = \text{const},\ \varphi = \text{const}: \quad d_z \vec{r} = \vec{e}_z \cdot dz \equiv \vec{k} \cdot dz$

- Linienelement

$$\vec{r} = \varrho \vec{e}_\varrho + z\vec{k}$$

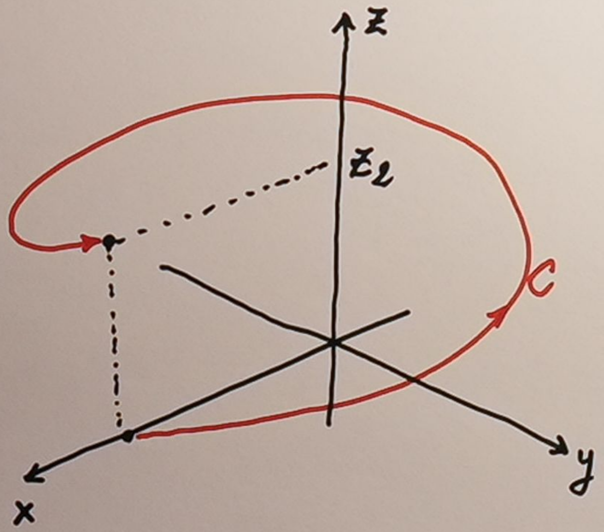
$$d\vec{r} = \vec{e}_\varrho \cdot d\varrho + \varrho \cdot d\vec{e}_\varrho + \vec{k} \cdot dz \quad , \quad d\vec{e}_\varrho = \vec{e}_\varphi \cdot d\varphi$$

$$= \vec{e}_\varrho \cdot d\varrho + \varrho \vec{e}_\varphi d\varphi + \vec{k} dz$$

$$= d_\varrho \vec{r} + d_\varphi \vec{r} + d_z \vec{r}$$

Linienelement: $\quad ds^2 = d\vec{r}\cdot d\vec{r} = d\varrho^2 + \varrho^2 d\varphi^2 + dz^2$

Beispiel: Bogenlänge einer Schraubenlinie



$$\vec{r} = R\cos\omega t\cdot\vec{i} + R\sin\omega t\cdot\vec{j} + ct\cdot\vec{k}\quad, \quad c = \text{const}$$

$$0 \le t \le \frac{2\pi}{\omega}$$

$$\varrho = R = \text{const}\ ,\quad d\varrho = 0$$

$$\varphi = \omega t$$

$$z = ct$$

$$s = \int_C ds = \int_C \sqrt{R^2 d\varphi^2 + dz^2}$$

$$= \int_0^{2\pi/\omega} \sqrt{R^2\dot{\varphi}^2 + \dot{z}^2}\,dt = \frac{2\pi}{\omega}\sqrt{R^2\omega^2 + c^2}$$

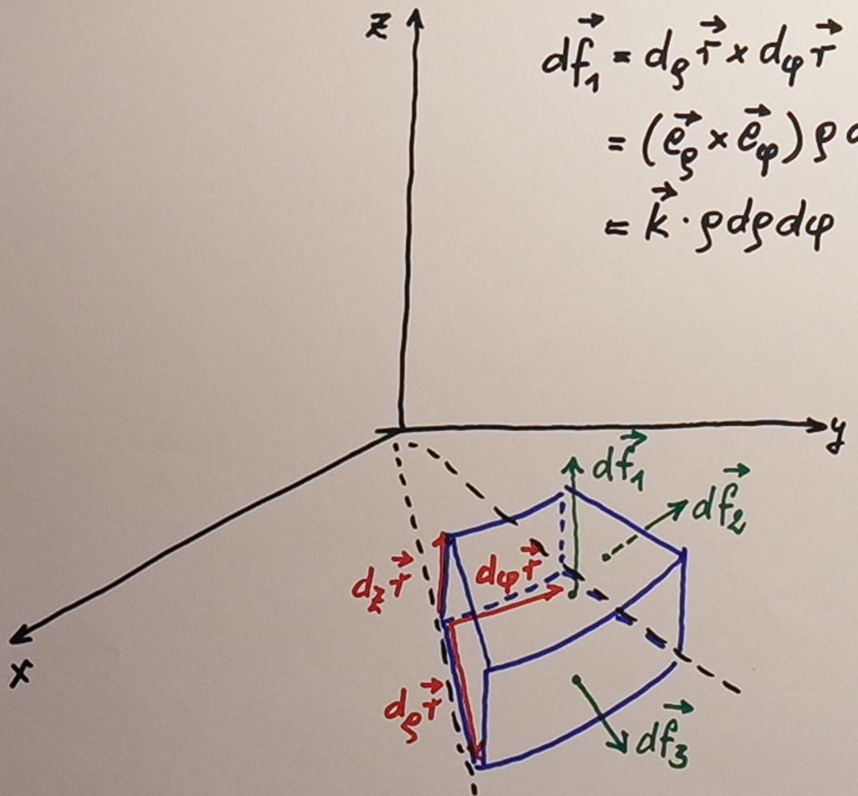
$$= \sqrt{4\pi^2 R^2 + z_2^2}$$

Diskussion:
- $c = 0: \quad s = 2\pi R$
- $c \gg R\omega: \quad s \approx z_2$
- $z = \frac{c}{\omega}\varphi$

Flächenelemente

– $z = const$

$$\vec{df_1} = d_\varrho \vec{r} \times d_\varphi \vec{r}$$
$$= (\vec{e}_\varrho \times \vec{e}_\varphi)\, \varrho\, d\varrho\, d\varphi$$
$$= \vec{k} \cdot \varrho\, d\varrho\, d\varphi$$



$\varphi = const$:

$$\vec{df_2} = d_z \vec{r} \times d_\varrho \vec{r}$$
$$= (\vec{k} \times \vec{e}_\varrho)\, d\varrho\, dz$$
$$= \vec{e}_\varphi \cdot d\varrho\, dz$$

$\varrho = const$:

$$\vec{df_3} = d_\varphi \vec{r} \times d_z \vec{r}$$
$$= (\vec{e}_\varphi \times \vec{k})\, \varrho\, d\varphi\, dz$$
$$= \vec{e}_\varrho \cdot \varrho\, d\varphi\, dz$$

Volumenelement

$$dV = (d_\varrho \vec{r} \times d_\varphi \vec{r}) \cdot d_z \vec{r}$$
$$= \vec{df_1} \cdot d_z \vec{r}$$

$$dV = d\varrho \cdot \varrho\, d\varphi \cdot dz$$

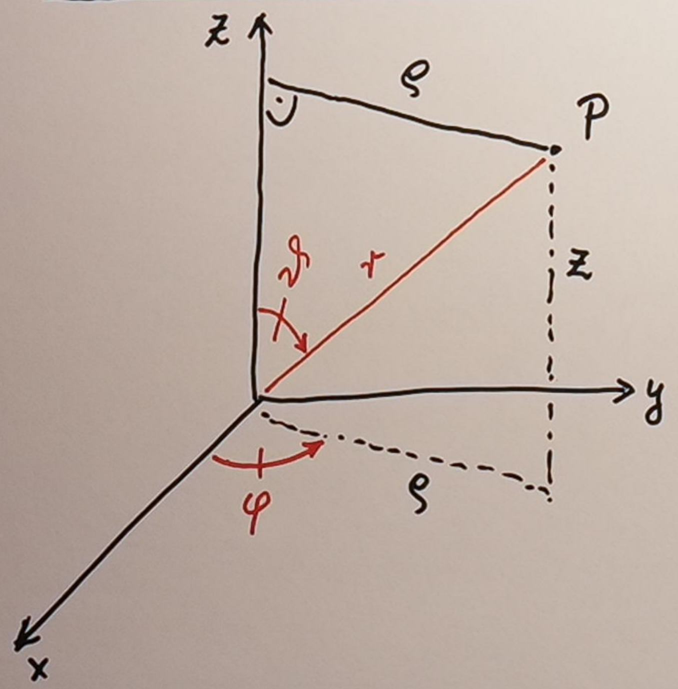
Beispiel: Zylindervolumen

$$V = \int_{Zyl.} dV = \int_0^R \varrho\, d\varrho \cdot \int_0^{2\pi} d\varphi \cdot \int_0^h dz$$
$$= \frac{R^2}{2} \cdot 2\pi \cdot h = \pi R^2 \cdot h$$

kartesisch: $$V = \iint_{Grundfl.} dx\, dy \cdot \int_0^h dz$$

Intermezzo: Krummlinige Orthogonalkoordinaten

III. Kugelkoordinaten (r, ϑ, φ)



$r \geq 0$

$0 \leq \vartheta \leq \pi$, Poldistanz

$(90° - \vartheta)$: geogr. Breite

$0 \leq \varphi < 2\pi$, Azimut

(geogr. Länge)

Zylinderkoordinaten:

$$x = \varrho \cos\varphi$$
$$y = \varrho \sin\varphi$$
$$z = z$$

$\downarrow$

$$\varrho = r \sin\vartheta$$
$$z = r\cos\vartheta$$

$\downarrow$

$$x = (r\sin\vartheta)\cos\varphi$$
$$y = (r\sin\vartheta)\sin\varphi$$
$$z = r\cos\vartheta$$

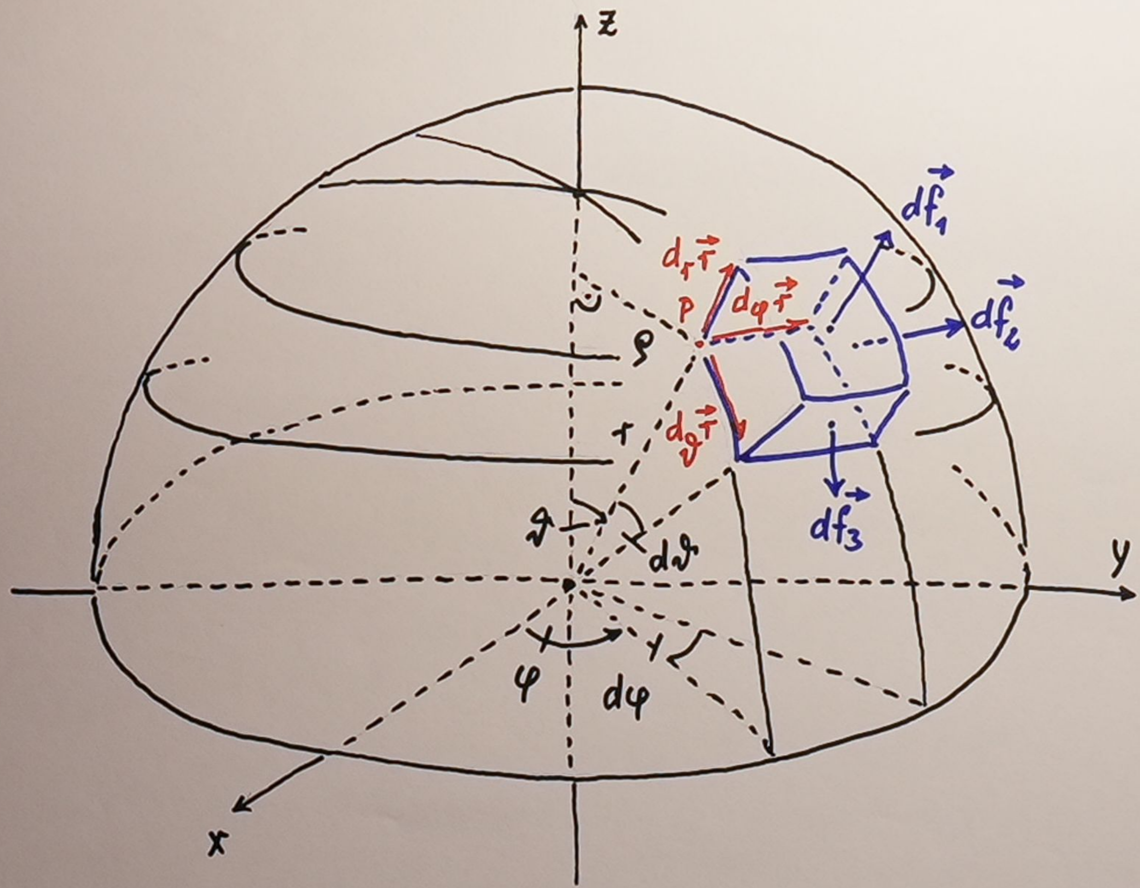
Kugelkoordinaten

Umkehrung: $r = \sqrt{x^2 + y^2 + z^2}$

$$\tan\varphi = \frac{y}{x}$$

$$\cos\vartheta = \frac{z}{r}$$

Tangentenvektoren der Koordinatenlinien



Ortsvektor von P: $\vec r = r\cdot\vec e_r(\vartheta,\varphi)$

Abstand zweier infinitesimal benachbarter Punkte:

$$d\vec r = \frac{\partial\vec r}{\partial r}dr + \frac{\partial\vec r}{\partial\vartheta}d\vartheta + \frac{\partial\vec r}{\partial\varphi}d\varphi$$

- radial ($\vartheta = const$, $\varphi = const$)

$$d_r\vec r = \frac{\partial\vec r}{\partial r}dr = \left(\frac{\partial x}{\partial r}\vec i + \frac{\partial y}{\partial r}\vec j + \frac{\partial z}{\partial r}\vec k\right)dr$$
$$= (\sin\vartheta\cos\varphi\,\vec i + \sin\vartheta\sin\varphi\,\vec j + \cos\vartheta\,\vec k)\,dr$$
$$= \vec e_r\,dr$$

- meridional ($r = const$, $\varphi = const$)

$$d_\vartheta\vec r = \frac{\partial\vec r}{\partial\vartheta}d\vartheta$$
$$= r\,(\cos\vartheta\cos\varphi\cdot\vec i + \cos\vartheta\sin\varphi\cdot\vec j - \sin\vartheta\cdot\vec k)\,d\vartheta$$
$$= r\cdot\vec e_\vartheta\,d\vartheta$$

- azimutal ($r = const$, $\vartheta = const$)

$$d_\varphi\vec r = \frac{\partial\vec r}{\partial\varphi}d\varphi = r\sin\vartheta\,(-\sin\varphi\cdot\vec i + \cos\varphi\,\vec j)\,d\varphi$$
$$= \vec e_\varphi\cdot r\sin\vartheta\,d\varphi$$

Tangenten-Einheitsvektoren:

$$\vec{e}_r = \sin\vartheta\,(\cos\varphi\cdot\vec{i} + \sin\varphi\cdot\vec{j}) + \cos\vartheta\cdot\vec{k}$$

$$\vec{e}_\vartheta = \cos\vartheta\,(\cos\varphi\cdot\vec{i} + \sin\varphi\cdot\vec{j}) - \sin\vartheta\,\vec{k}$$

$$\vec{e}_\varphi = -\sin\varphi\cdot\vec{i} + \cos\varphi\cdot\vec{j}$$

Richtungsänderungen:

$$\frac{\partial\vec{e}_r}{\partial\vartheta} = \vec{e}_\vartheta\,, \quad \frac{\partial\vec{e}_r}{\partial\varphi} = \sin\vartheta\cdot\vec{e}_\varphi$$

$$\frac{\partial\vec{e}_\vartheta}{\partial\vartheta} = -\vec{e}_r\,, \quad \frac{\partial\vec{e}_\vartheta}{\partial\varphi} = \cos\vartheta\cdot\vec{e}_\varphi$$

$$\frac{\partial\vec{e}_\varphi}{\partial\varphi} = -\left(\vec{e}_r\sin\vartheta + \vec{e}_\vartheta\cos\vartheta\right)$$

Linienelement:

$$d\vec{r} = \vec{e}_r\cdot dr + \vec{e}_\vartheta\cdot r\,d\vartheta + \vec{e}_\varphi\cdot r\sin\vartheta\,d\varphi$$

$$\underline{ds^2} = d\vec{r}\cdot d\vec{r} = \underline{dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2}$$

Flächenelemente:

- $r = \text{const}$: $d\vec{f}_1 = d_\vartheta\vec{r}\times d_\varphi\vec{r} = \vec{e}_r\cdot r^2\sin\vartheta\,d\vartheta\,d\varphi$

  Kugeloberfläche

  Beispiel: Kugel, Radius R

$$\underline{\underline{A}} = R^2\int_0^\pi \sin\vartheta\,d\vartheta\int_0^{2\pi} d\varphi = R^2(-\cos\vartheta)\Big|_0^\pi\cdot 2\pi = \underline{\underline{4\pi R^2}}$$

- $\varphi = \text{const}$: $d\vec{f}_2 = d_r\vec{r}\times d_\vartheta\vec{r} = \vec{e}_\varphi\cdot r\,dr\,d\vartheta$

  Meridiankreis

- $\vartheta = \text{const}$: $d\vec{f}_3 = d_\varphi\vec{r}\times d_r\vec{r} = \vec{e}_\vartheta\cdot r\sin\vartheta\,dr\,d\varphi$

  Mantelfläche eines Kreiskegels

Beispiel:

$$A = \sin\vartheta_0\int_0^R r\,dr\int_0^{2\pi} d\varphi = \pi R\,(R\sin\vartheta_0)$$

Volumenelement:

Spatprodukt: $(d_\vartheta \vec{r} \times d_\varphi \vec{r}) \cdot d_r \vec{r} = d\vec{f}_1 \cdot d_r \vec{r}$

$$\underline{dV = r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi}$$

in Abbildung: $dV = (r\,d\vartheta \cdot r\sin\vartheta\,d\varphi) \cdot dr$

Beispiel: Kugelvolumen, Radius R

$$V = \int_0^R r^2 dr \underbrace{\int_0^\pi \sin\vartheta \, d\vartheta \int_0^{2\pi} d\varphi}_{4\pi}$$

$$\underline{\underline{V}} = \frac{R^3}{3} \cdot 4\pi = \underline{\underline{\frac{4\pi}{3} R^3}}$$

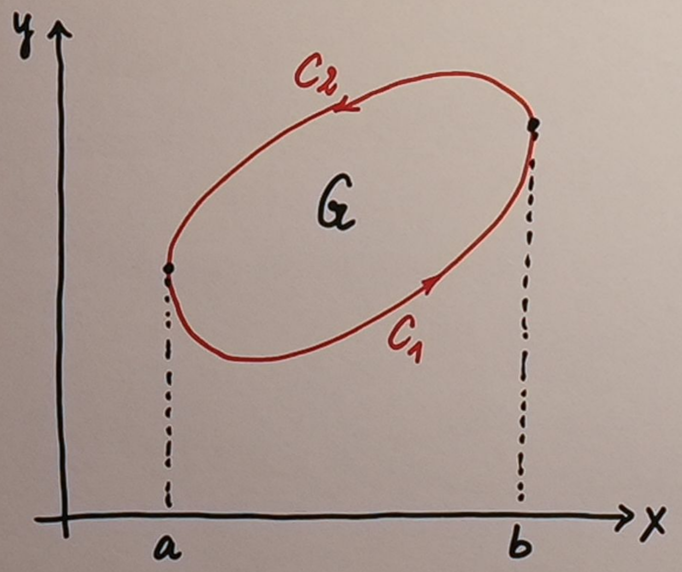
Der Greensche Satz

I. Herleitung des Greenschen Satzes

2-dim. – $\vec{\Phi} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

– $\oint_C \vec{\Phi}\, d\vec{r} = \oint_C (P\,dx + Q\,dy)$

1. Schritt $\oint_C P(x,y)\,dx$



C_1: $y = f_1(x)$

C_2: $y = f_2(x)$

$$\oint_C P(x,y)\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx$$

$$= \int_a^b P[x, f_1(x)]\,dx + \int_b^a P[x, f_2(x)]\,dx$$

$$= \int_a^b \{P[x, f_1(x)] - P[x, f_2(x)]\}\,dx$$

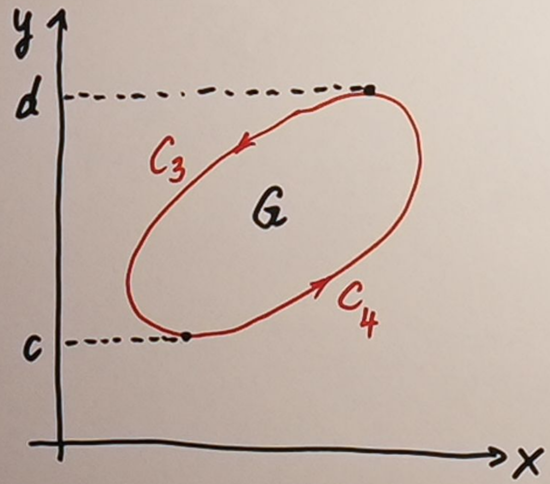
$$\iint_G \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}\,dy\right] dx$$

$$= \int_a^b P(x,y)\Big|_{y=f_1(x)}^{f_2(x)}\,dx$$

$$= \int_a^b \{P[x, f_2(x)] - P[x, f_1(x)]\}\,dx$$

$$\longrightarrow \oint_C P\,dx = -\iint_G \frac{\partial P}{\partial y}\,dx\,dy$$

2. Schritt



$C_3: \quad x = f_3(y)$

$C_4: \quad x = f_4(y)$

$$\oint_C Q(x,y)\,dy = \int_c^d Q[f_4(y),y]\,dy + \int_d^c Q[f_3(y),y]\,dy$$

$$= \int_c^d \{Q[f_4(y),y] - Q[f_3(y),y]\}\,dy$$

---

andererseits:

$$\iint_G \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d \left[\int_{f_3(y)}^{f_4(y)} \frac{\partial Q}{\partial x}\,dx\right] dy$$

$$= \int_c^d \{Q[f_4(y),y] - Q[f_3(y),y]\}\,dy$$

$$\longrightarrow \quad \oint_C Q\,dy = \iint_G \frac{\partial Q}{\partial x}\,dx\,dy$$

Schritte 1 und 2 zusammen: <u>Greenscher Satz</u>

$$\oint_C \vec{\phi}\,d\vec{r} = \oint_C (P\,dx + Q\,dy) = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

II. Beispiele

1.) Konservatives Kraftfeld

$$P = F_x\,,\quad Q = F_y$$

$$\oint_C \vec{F}\,d\vec{r} = \oint_C (F_x\,dx + F_y\,dy)$$

$$= \iint_G \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\,dy$$

für $\dfrac{\partial F_y}{\partial x} = \dfrac{\partial F_x}{\partial y} \longrightarrow \oint_C \vec{F}\,d\vec{r} = 0$

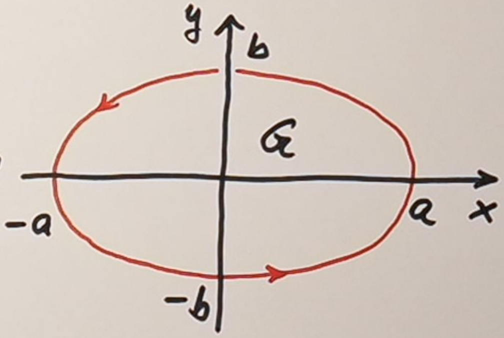
2.) Flächeninhalt als Linienintegral

$$P = -y\,,\quad Q = x$$

$$\oint_C (x\,dy - y\,dx) = 2\iint_G dx\,dy$$

$$\longrightarrow \quad A_G = \frac{1}{2}\oint_C (x\,dy - y\,dx)$$

Beispiel: Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



$\rightarrow C:\ y = \pm \dfrac{b}{a}\sqrt{a^2 - x^2}$

• Flächenintegral

$$\int_{-a}^{a}\left[\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\right] dx = 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4\frac{b}{a}\cdot\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a}\right)\Bigg|_0^a$$

$$= 2\frac{b}{a}\cdot a^2 \arcsin 1 = 2ab\cdot\frac{\pi}{2}$$

$$= \pi a b$$

- Kurvenintegral

$$y = +\frac{b}{a}\sqrt{a^2-x^2}$$

$$dy = -\frac{b}{a}\frac{x\,dx}{\sqrt{a^2-x^2}}$$

$$\frac{1}{2}\oint_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\frac{b}{a}\int_{-a}^{a}\left(\frac{x^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}\right)dx - \frac{1}{2}\frac{b}{a}\int_{a}^{-a}\left(\frac{x^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}\right)dx$$

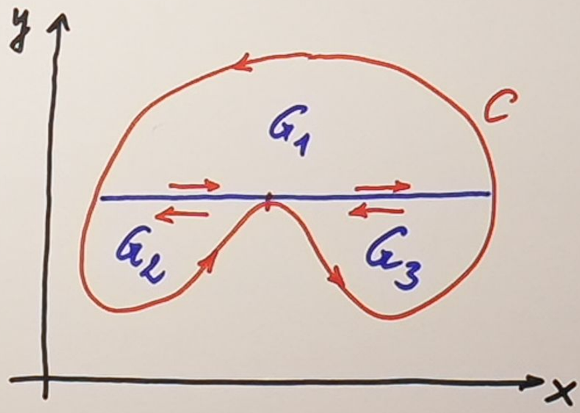
unterer Ellipsenbogen   oberer Ellipsenbogen

$$= \frac{b}{a}\int_{-a}^{a}\left(\frac{x^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}\right)dx$$

$$= ab\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}} = 2ab\cdot\arcsin\frac{x}{a}\Big|_0^a$$

$$= 2ab\cdot\frac{\pi}{2} = \pi ab$$

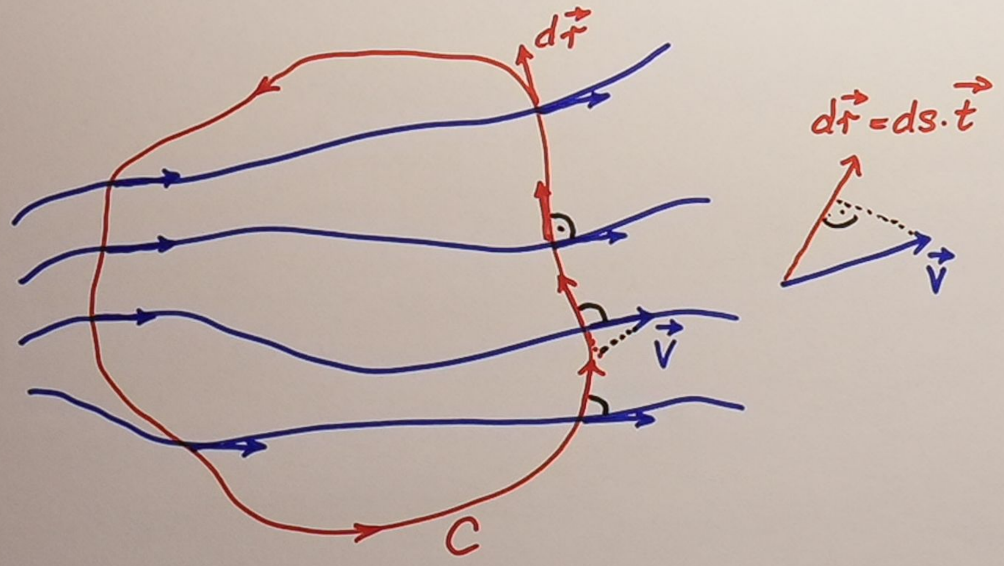
3.) komplizierte Gebiete G



Der Stokessche Satz. Zirkulation und Rotation

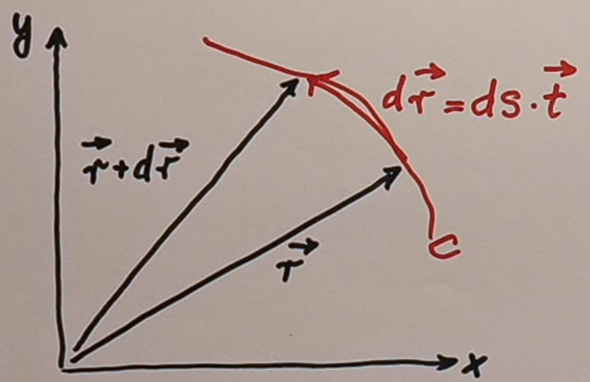
I. Zirkulation eines Vektorfeldes

Greenscher Satz, $\vec{\Phi} \to \vec{v}$: Strömungs-, Geschwindigkeitsfeld



$$\oint_C \vec{v}\, d\vec{r}$$

$$= \oint_C (\vec{v} \cdot \vec{t})\, ds$$



$$\oint_C \vec{v}\, d\vec{r}$$

> 0 , $\sphericalangle(\vec{v}, \vec{t})$ überwiegend spitz

Strömung „netto" <u>entgegen</u> Uhrzeigersinn

$= 0$, keine Netto-Umströmung der Kurve

< 0 , $\sphericalangle(\vec{v}, \vec{t})$ überwiegend stumpf

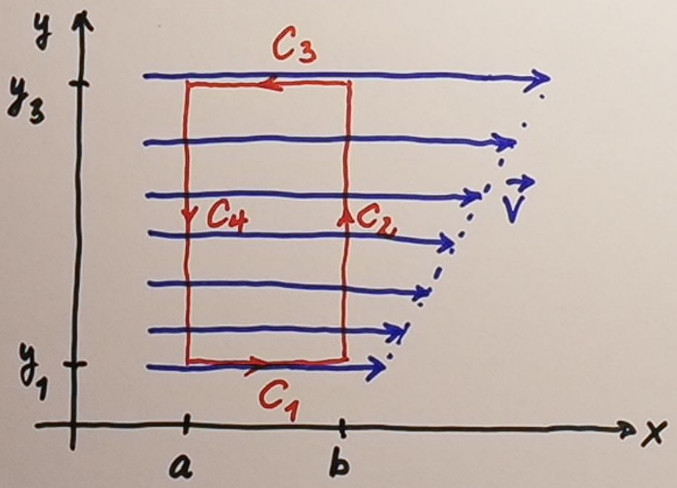
Strömung „netto" <u>im</u> Uhrzeigersinn

$\oint_C (\vec{v} \cdot \vec{t})\, ds$ Zirkulation von $\vec{v}$ bzgl. C (Umströmung)

$\vec{\Phi}$	Zirkulation
$\vec{F}$	Arbeit
$\vec{E}$	Umlaufspannung

Beispiele für ebene Strömungen

1.) Lineare Strömung $\vec{v} = \alpha y \cdot \vec{i}\,, \quad \alpha > 0$



Zirkulation: $\oint_C \vec{v}\,d\vec{r}$

$$= \int_{C_1} \vec{v}\,d\vec{r} + \int_{C_2} \vec{v}\,d\vec{r} + \int_{C_3} \vec{v}\,d\vec{r} + \int_{C_4} \vec{v}\,d\vec{r}$$

C_1	C_2	C_3	C_4
$a \le x \le b$, $y = y_1$	$dx = 0$	$b \ge x \ge a$, $y = y_3$	$dx = 0$

$$= \alpha y_1 \int_a^b dx + 0 + \alpha y_3 \int_b^a dx + 0$$

$$= \alpha (y_1 - y_3)(b-a) < 0$$

Flächenintegral (Green)

$$P(x,y) = \alpha y\,, \quad Q(x,y) = 0$$

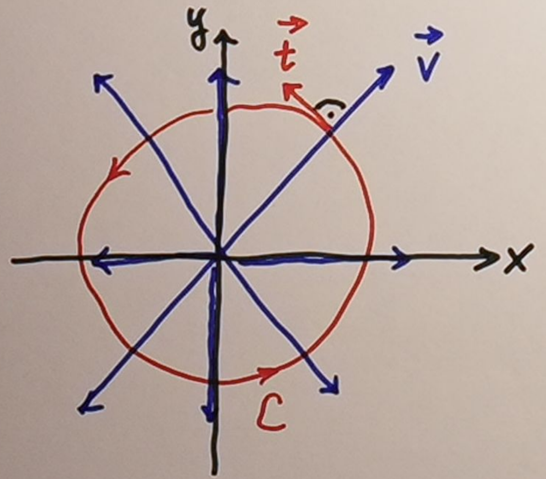
$$\frac{\partial P}{\partial y} = \alpha\,, \quad \frac{\partial Q}{\partial x} = 0$$

$$\iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = -\alpha \iint_G dx\,dy$$

$$= -\alpha \cdot A_G = \alpha (b-a)(y_1 - y_3)$$

2.) Radiale Strömung

$$\vec{v} = \alpha \vec{r} = \alpha(x\vec{i} + y\vec{j}) \, , \quad \alpha \begin{cases} >0: \text{vom Zentrum weg} \\ <0: \text{zum Zentrum hin} \end{cases}$$



Zirkulation: $\sphericalangle(\vec{v}, \vec{t}) = \frac{\pi}{2}$

$$\oint_C \vec{v}\, d\vec{r} = 0$$

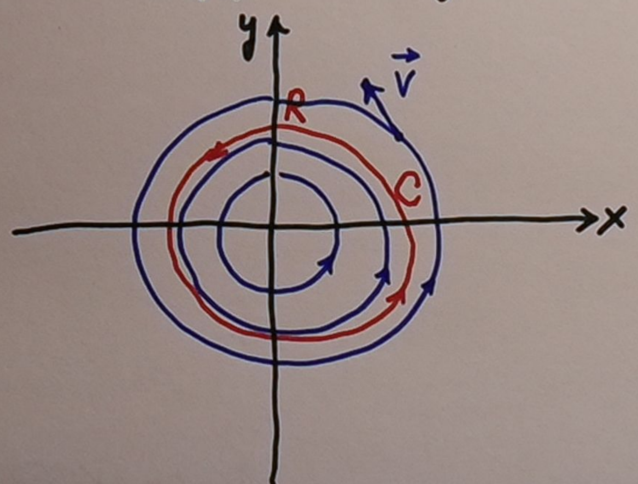
Flächenintegral: $P = \alpha x$, $Q = \alpha y$

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$$

$$\iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = 0$$

3.) Starre Rotation $\vec{v} = \omega(-y\vec{i} + x\vec{j}) = \vec{\omega} \times \vec{r}$

entgegen Uhrzeigersinn



Zirkulation: $\sphericalangle(\vec{v}, \vec{t}) = 0$

$$\oint_C \vec{v}\, d\vec{r} = \oint_C v \cdot ds$$

$$= \oint_C \omega r \cdot r\, d\varphi = \omega R^2 \int_0^{2\pi} d\varphi$$

$(r = R)$

$$= 2\pi \omega R^2 > 0$$

Flächenintegral: $P = -\omega y$, $Q = \omega x$

$$\frac{\partial P}{\partial y} = -\omega \, , \quad \frac{\partial Q}{\partial x} = \omega$$

$$\iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

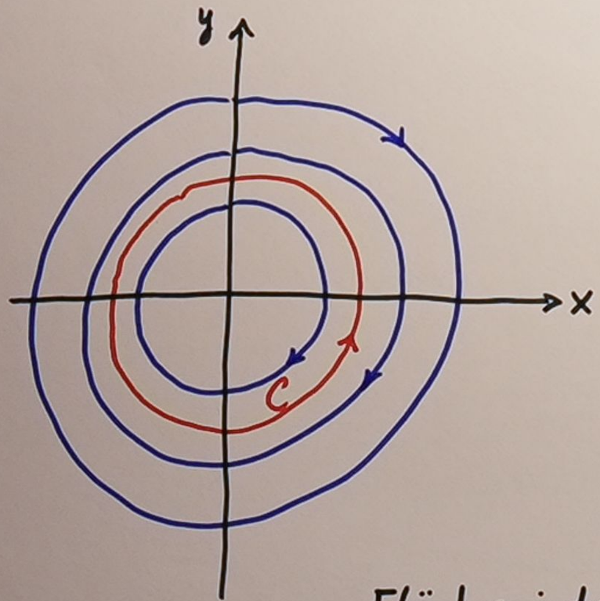
$$= 2\omega \iint_G dx\, dy = 2\omega \cdot A_G$$

$$= 2\omega \cdot \pi R^2$$

4.) „Abfluß" im Uhrzeigersinn

$$\vec{v} = \frac{\alpha y}{r^2}\vec{i} - \frac{\alpha x}{r^2}\vec{j}\ , \quad \alpha > 0$$

$$v = \frac{\alpha}{r}$$



Zirkulation: $\sphericalangle(\vec{v}, \vec{t}) = \pi$

$$\oint_C \vec{v}\, d\vec{r} = -\oint_C v\, ds$$

$$= -\int_0^{2\pi} \frac{\alpha}{r} \cdot r\, d\varphi$$

$$= -\alpha \cdot 2\pi < 0$$

Flächenintegral: $P = \frac{\alpha y}{r^2}$, $Q = -\frac{\alpha x}{r^2}$

$$\frac{\partial P}{\partial y} = \frac{\alpha}{r^2}\left(1 - 2\frac{y^2}{r^2}\right), \quad \frac{\partial Q}{\partial x} = -\frac{\alpha}{r^2}\left(1 - 2\frac{x^2}{r^2}\right)$$

$$\iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = 2\alpha \iint_G \left(-\frac{1}{r^2} + \frac{x^2 + y^2}{r^4}\right) dx\, dy = 0$$

Der Stokessche Satz. Zirkulation und Rotation

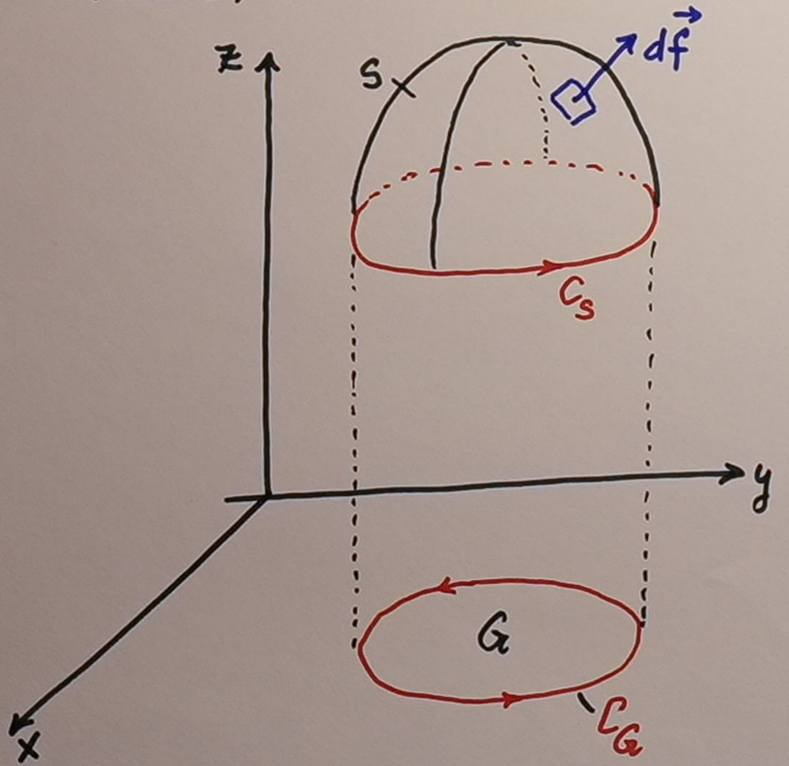
II. Verallgemeinerung des Greenschen Satzes auf drei Dimensionen

- Vektorfeld: $\vec{\Phi} = P\vec{i} + Q\vec{j} \longrightarrow \vec{\Phi} = P \cdot \vec{i} + Q\vec{j} + R\vec{k}$

$$P = P(x, y, z)$$

$$Q, R \ldots$$

- Fläche, Randkurve



$S: z = z(x, y)$

Zirkulation von $\vec{\Phi}$ bzgl. C_S:

$$\oint_{C_S} \vec{\Phi}\, d\vec{r} = \oint_{C_S} (P dx + Q dy + R dz)$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\oint_{C_S} \vec{\Phi}\, d\vec{r} = \oint_{C_G} \left[\left(P + R \frac{\partial z}{\partial x}\right) dx + \left(Q + R \frac{\partial z}{\partial y}\right) dy \right]$$

Anwendung des Greenschen Satzes:

$$P \to P + R\frac{\partial z}{\partial x}$$

$$Q \to Q + R\frac{\partial z}{\partial y}$$

$$\oint_{C_s} \vec{\Phi}\, d\vec{r} = \iint_G \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dx\,dy$$

$$P = P[x, y, z(x,y)] \;,\quad Q, R \ldots$$

$$= \iint_G \Big[\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \cancel{\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}} + \cancel{R\frac{\partial^2 z}{\partial x \partial y}}$$

$$- \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} - \cancel{\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}} - \cancel{R\frac{\partial^2 z}{\partial y \partial x}} \Big] dx\,dy$$

$$= \iint_G \left[\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\frac{\partial z}{\partial x} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dx\,dy$$

$$= \iint_G \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right]\left[-\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k}\right] dx\,dy$$

Diskussion der beiden Faktoren des Skalarprodukts

1. Faktor $\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i}-\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k} \equiv \text{rot}\,\vec{\phi}$ „Rotation"

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{\nabla}\times\vec{\phi} \quad , \quad \boxed{\vec{\nabla}=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}}$$

„Nabla-Operator"

$\left(\vec{\nabla}u(x,y,z)\equiv \text{grad}\,u\right)$

$$\text{rot}\,\vec{\phi}=\vec{\nabla}\times\vec{\phi} \quad (=\text{curl}\,\vec{\phi})$$

2. Faktor $-\frac{\partial z}{\partial x}\vec{i}-\frac{\partial z}{\partial y}\vec{j}+\vec{k}$

Normalenvektor von S: $z=z(x,y)$

$$d\vec{f}=\left(-\frac{\partial z}{\partial x}\vec{i}-\frac{\partial z}{\partial y}\vec{j}+\vec{k}\right)dx\,dy$$

Resultat:
$$\boxed{\oint_{C_s}\vec{\phi}\,d\vec{r}=\iint_S \text{rot}\,\vec{\phi}\,d\vec{f}}$$
Stokesscher Integralsatz

Anmerkung:

$$\oiint_S \text{rot}\,\vec{\phi}\,d\vec{f}=0$$

Der Stokessche Satz. Zirkulation und Rotation

III. Zur Bedeutung der Rotation

1. Beispiel: Lineare Strömung

$$\vec{v} = \alpha \cdot y \cdot \vec{i}\,, \quad \alpha > 0$$

Zirkulation: $\oint_C \vec{v}\, d\vec{r} = -\alpha A_G$

Rotation:

$$\operatorname{rot} \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha y & 0 & 0 \end{vmatrix} = -\alpha \vec{k}$$

$$d\vec{f} = \vec{k} \cdot dx\,dy$$

$$\iint_S \operatorname{rot} \vec{v}\, d\vec{f} = -\alpha \iint_G dx\,dy = -\alpha A_G$$

2. Beispiel: Radiale Strömung $\vec{v} = \alpha \vec{r}$

Zirkulation: $\oint_C \vec{v}\, d\vec{r} = 0$

Rotation:

$$\operatorname{rot} \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha x & \alpha y & 0 \end{vmatrix} = \vec{0}\,, \quad \text{wirbelfrei}$$

$$d\vec{f} = \vec{k} \cdot dx\,dy$$

$$\iint_S \operatorname{rot} \vec{v}\, d\vec{f} = 0$$

Anmerkung: $\operatorname{rot} \vec{r} = \vec{0}$ auch in 3 Dimensionen

3. Beispiel: Starre Rotation

$$\vec{v} = \vec{\omega} \times \vec{r}\,, \quad \vec{\omega} = \omega \vec{k}$$

Zirkulation: $\oint_C \vec{v}\, d\vec{r} = 2\pi\omega R^2 = 2\omega \cdot A_G$

Rotation:

$$\operatorname{rot} \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \vec{k}$$

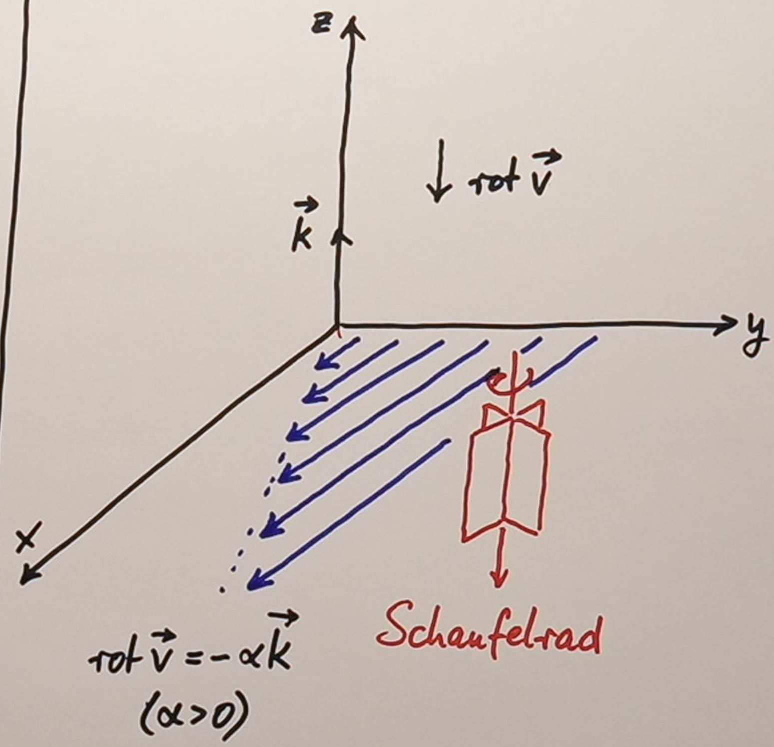
$$d\vec{f} = \vec{k} \cdot dx\, dy$$

$$\iint_S \operatorname{rot} \vec{v}\, d\vec{f} = 2\omega \iint_G dx\, dy = 2\omega \cdot A_G$$

Fazit: $\vec{k} \cdot \operatorname{rot} \vec{v} = \frac{1}{A_G} \oint_C \vec{v}\, d\vec{r}$

Normalkomponente der Rotation ist Zirkulation pro Fläche.

$\vec{k} \cdot \operatorname{rot} \vec{v}$ maximal für $\vec{k} \parallel \operatorname{rot} \vec{v}$



Fazit: $\operatorname{rot} \vec{v}$ zeigt Richtung an, in die man ein Schaufelrad halten muß, damit es sich, von der Spitze seiner Achse gesehen, entgegen dem Uhrzeigersinn am schnellsten dreht.

Der Stokessche Satz. Zirkulation und Rotation

IV. Beispiele für den Stokesschen Satz

1. Beispiel: Konservative Kraft, $\vec{\Phi} \to \vec{F}$

falls $\operatorname{rot} \vec{F} = \vec{0} \longrightarrow$ Stokescher Satz:

$$\oint_C \vec{F}\,d\vec{r} = 0 \,, \quad \vec{F} \text{ konservativ}$$

$$\vec{F} = -\operatorname{grad} U(x,y,z)$$

$$\operatorname{rot}\operatorname{grad} U = \vec{\nabla} \times (\vec{\nabla} U) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$

$$= \vec{i} \underbrace{\left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y} \right)}_{=0} - + \cdots$$

$$\operatorname{rot}\operatorname{grad} U = \vec{0}$$

Fazit: $\operatorname{rot} \vec{F} = \vec{0}$, $\vec{F}$ konservativ

2. Beispiel:

- Vektorfeld:

$$\vec{\Phi} = (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$$
$$= \vec{r} \times (\vec{i} + \vec{j} + \vec{k})$$

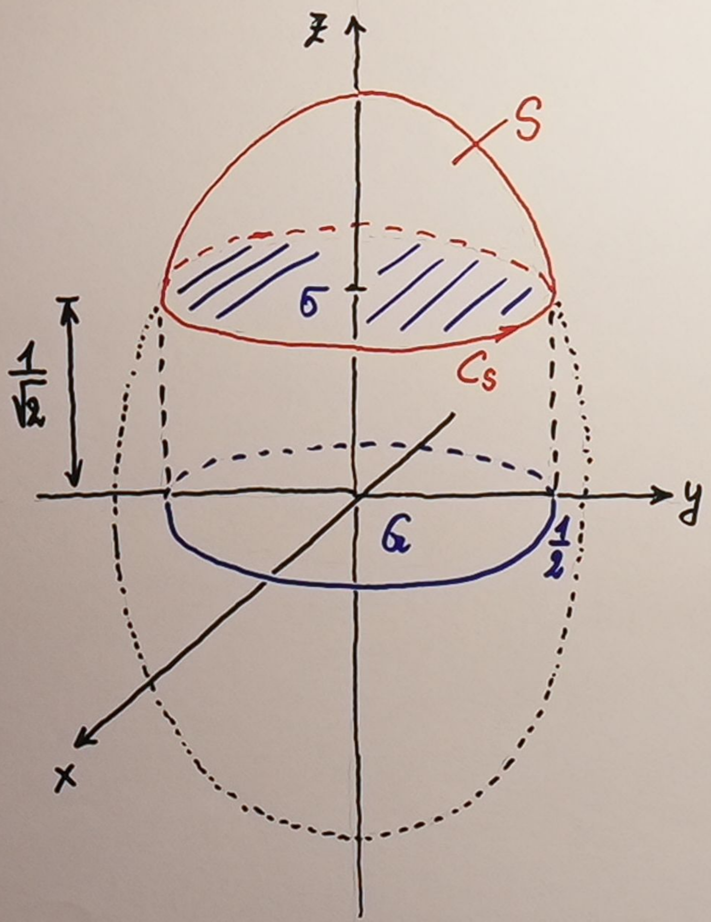
$$\longrightarrow \quad \operatorname{rot} \vec{\Phi} = -2(\vec{i} + \vec{j} + \vec{k})$$

- Fläche S: Kappe eines Rotationsellipsoids

Rotationsellipsoid $2(x^2+y^2) + z^2 = 1$

Randkurve C_S: Kreis in Höhe $z = \frac{1}{\sqrt{2}}$

$$2(x^2+y^2) + \frac{1}{2} = 1$$
$$x^2 + y^2 = \frac{1}{4} \,, \quad \text{Radius } \frac{1}{2}$$



C_s: $\quad x^2+y^2=\frac{1}{4}$

Polarkoordinaten: $\quad x=\frac{1}{2}\cos\varphi$

$$y=\frac{1}{2}\sin\varphi$$

$$\vec{r}=\frac{1}{2}\cos\varphi\cdot\vec{i}+\frac{1}{2}\sin\varphi\cdot\vec{j}+\frac{1}{\sqrt{2}}\vec{k}$$

$$d\vec{r}=\left(-\frac{1}{2}\sin\varphi\cdot\vec{i}+\frac{1}{2}\cos\varphi\cdot\vec{j}\right)d\varphi$$

Zirkulation

$$\oint_{C_s}\vec{\Phi}\,d\vec{r}=\int_0^{2\pi}\left[-\frac{1}{2}\left(\frac{1}{2}\sin\varphi-\frac{1}{\sqrt{2}}\right)\sin\varphi+\frac{1}{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\cos\varphi\right)\cos\varphi\right]d\varphi$$

$$=-\frac{1}{4}\int_0^{2\pi}\underbrace{\left(\sin^2\varphi+\cos^2\varphi\right)}_{=1}d\varphi$$

$$+\frac{1}{2\sqrt{2}}\underbrace{\int_0^{2\pi}\sin\varphi\,d\varphi}_{=0}+\frac{1}{2\sqrt{2}}\underbrace{\int_0^{2\pi}\cos\varphi\,d\varphi}_{=0}=-\frac{1}{4}\cdot 2\pi=\underline{\underline{-\frac{\pi}{2}}}$$

Oberflächenintegral

$$S:\quad z=\sqrt{1-2(x^2+y^2)}$$

$$d\vec{f}=\left(-\frac{\partial z}{\partial x}\vec{i}-\frac{\partial z}{\partial y}\vec{j}+\vec{k}\right)dx\,dy=\left(\frac{2x}{z}\vec{i}+\frac{2y}{z}\vec{j}+\vec{k}\right)dx\,dy$$

$$\longrightarrow\qquad \mathrm{rot}\,\vec{\Phi}\cdot d\vec{f}=-2\,\frac{2x+2y+z}{z}\,dx\,dy$$

Polarkoordinaten auf G

$$\iint\limits_S \operatorname{rot}\vec{\phi}\, d\vec{f} = -2 \iint\limits_G \left[\frac{2(x+y)}{\sqrt{1-2(x^2+y^2)}} + 1 \right] dx\,dy$$

$$= -2 \int_0^{1/2} r\,dr \int_0^{2\pi} d\varphi \left[\frac{2r(\cos\varphi + \sin\varphi)}{\sqrt{1-2r^2}} + 1 \right]$$

$$= -4 \int_0^{1/2} \frac{r^2\,dr}{\sqrt{1-2r^2}} \left[\underbrace{\int_0^{2\pi} \cos\varphi\, d\varphi}_{=0} + \underbrace{\int_0^{2\pi} \sin\varphi\, d\varphi}_{=0} \right]$$

$$-2 \int_0^{1/2} r\,dr \int_0^{2\pi} d\varphi$$

$$= -2 \left. \frac{r^2}{2} \right|_0^{1/2} \cdot 2\pi = -\frac{\pi}{2}$$

Deformation der Fläche $S \longrightarrow \sigma$

$$\sigma: \; x^2 + y^2 \leq \frac{1}{4}, \quad z = \frac{1}{\sqrt{2}}$$

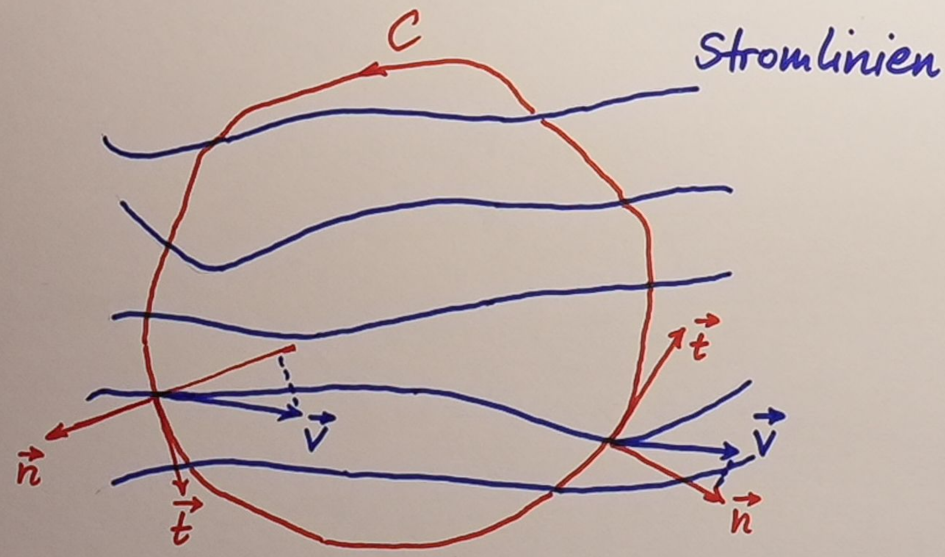
$$d\vec{f} = \vec{k} \cdot dx\,dy$$

$$\iint\limits_\sigma \operatorname{rot}\vec{\phi}\, d\vec{f} = -2 \iint\limits_G dx\,dy$$

$$= -2 \cdot \frac{\pi}{4} = -\frac{\pi}{2}$$

Der Gaußsche Satz. Fluß und Divergenz

I. Fluß eines Vektorfeldes und Gaußscher Satz für die Ebene



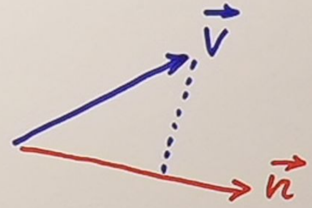
Stokes: $\oint_C \vec{v}\cdot d\vec{r} = \oint_C (\vec{v}\cdot\vec{t})\,ds$

Zirkulation

<u>Um</u>strömung von C

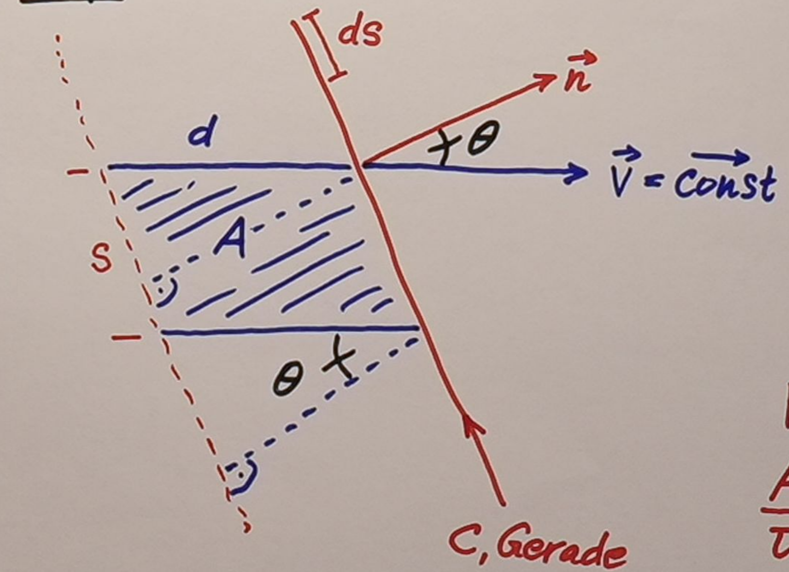
Tangentialkomponente von $\vec{v}$

Gauß: $\oint_C (\vec{v}\cdot\vec{n})\,ds$



<u>Durch</u>strömung, Normalkomponente von $\vec{v}$

<u>Fluß</u>



Durchflußzeit

$$\tau = \frac{d}{v} = \frac{A}{s\cdot v\cdot\cos\theta}$$

Fluß:

$$\frac{A}{\tau} = s\cdot v\cdot\cos\theta = s\cdot(\vec{v}\cdot\vec{n})$$

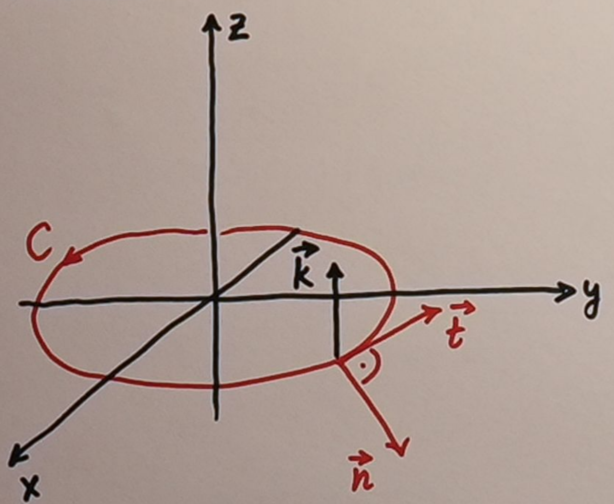
Maß für Flüssigkeitsmenge $\quad A = s\cdot d\cdot\cos\theta$

Verallgemeinerung: $\oint_C (\vec{v}\cdot\vec{n})\,ds$ Fluß

$$\vec{v} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

Fluß: $\oint_C (P\vec{i} + Q\vec{j})\,\vec{n}\,ds$

<u>Normalenvektor</u>



$$\vec{n} = \vec{t}\times\vec{k}$$

$$= \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left[\dot{x}(\vec{i}\times\vec{k}) + \dot{y}(\vec{j}\times\vec{k})\right]$$

$$\vec{n} = \frac{\dot{y}\,\vec{i} - \dot{x}\,\vec{j}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

C: Parameter t

$$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{\dot{x}^2+\dot{y}^2}\,dt$$

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt}\cdot\vec{t}$$

$$\vec{t} = \frac{\dot{x}\,\vec{i} + \dot{y}\,\vec{j}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

Fluß: $\oint_C (\vec{v}\cdot\vec{n})\,ds$

$$= \oint_C (P\vec{i}+Q\vec{j})\,\frac{\dot{y}\,\vec{i} - \dot{x}\,\vec{j}}{\sqrt{\dot{x}^2+\dot{y}^2}}\,\sqrt{\dot{x}^2+\dot{y}^2}\,dt$$

$$= \oint_C (P\vec{i}+Q\vec{j})\left(\frac{dy}{dt}\,\vec{i} - \frac{dx}{dt}\,\vec{j}\right)dt$$

$$= \oint_C (P\,dy - Q\,dx)$$

Green: $P \rightarrow -Q$
$Q \rightarrow P$

$$= \iint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dx\,dy$$

Gaußscher Satz für die Ebene

Stokes:

$$\oint_C (\vec{v}\cdot\vec{t})\,ds = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Der Gaußsche Satz. Fluß und Divergenz

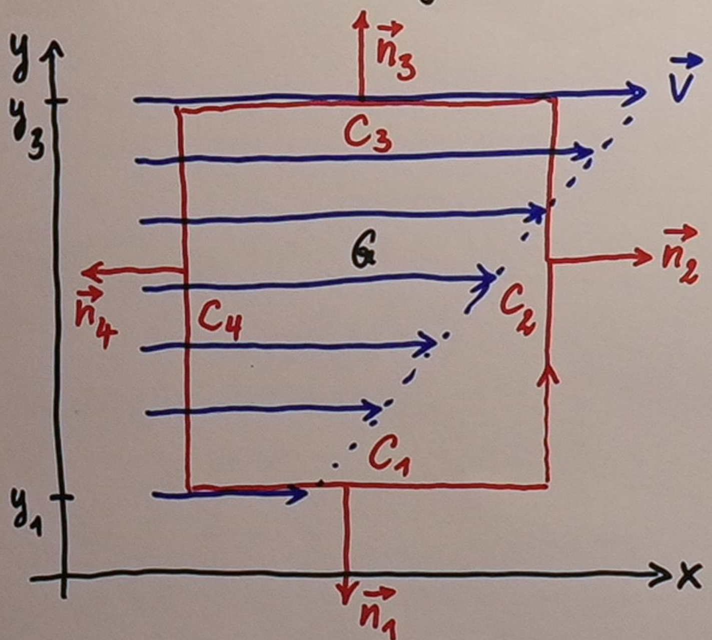
II. Beispiele für ebene Strömungen

$$\oint_C (\vec{v}\cdot\vec{n})\,ds = \oint_C (P\,dy - Q\,dx)$$

$$= \iint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy$$

Beispiel 1 Lineare Strömung $\vec{v} = \alpha y \cdot \vec{i}$, $\alpha > 0$

$\longrightarrow P = \alpha y$, $Q = 0$



	$\vec{n}$	$\vec{v}\vec{n}$	ds
C_1	$-\vec{j}$	0	dx
C_2	$\vec{i}$	αy	dy
C_3	$\vec{j}$	0	$-dx$
C_4	$-\vec{i}$	$-\alpha y$	$-dy$

Fluß: $$\oint_C (\vec{v}\cdot\vec{n})\,ds = \int_{y_1}^{y_3} \alpha y \cdot dy + \int_{y_3}^{y_1} (-\alpha y)\cdot(-dy)$$

$$= 0$$

Flächenintegral: $\dfrac{\partial P}{\partial x} = 0$, $\dfrac{\partial Q}{\partial y} = 0$

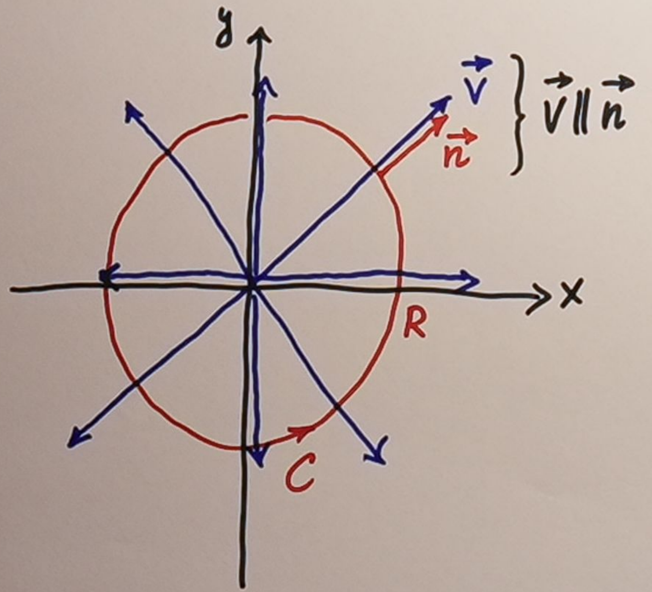
$$\iint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = 0$$

Strömung „inkompressibel“

Beispiel 2 Radiale Strömung

$$\vec{v} = \alpha \vec{r} = \alpha(x\vec{i} + y\vec{j})$$

$$P = \alpha x, \quad Q = \alpha y$$



Fluß: $\vec{v} \parallel \vec{n} \longrightarrow \cos\sphericalangle(\vec{v},\vec{n}) = 1$

$$\vec{v}\cdot\vec{n} = v \cdot \cos\sphericalangle(\vec{v},\vec{n}) = v = \alpha r$$

auf C: $r = R$

$$\oint_C (\vec{v}\cdot\vec{n})\,ds = \int_0^{2\pi} \alpha R \cdot R\,d\varphi$$

$$= 2\pi\alpha \cdot R^2$$

Flächenintegral: $\dfrac{\partial P}{\partial x} = \alpha = \dfrac{\partial Q}{\partial y}$

$$\iint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = 2\alpha \iint_G dx\,dy$$

$$= 2\alpha \cdot \pi R^2, \quad \text{Übereinstimmung}$$

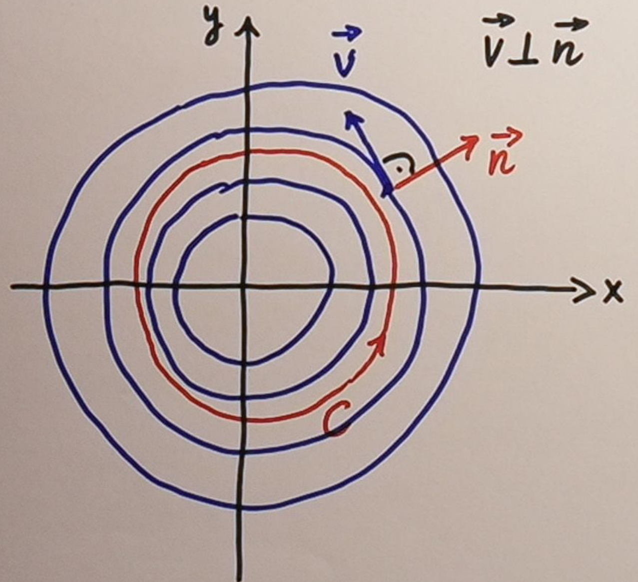
Resultat:

$$\oint_C (\vec{v}\cdot\vec{n})\,ds = 2\pi\alpha R^2 \begin{cases} > 0, & \alpha > 0 \quad \text{Strömung expandiert} \\ < 0, & \alpha < 0 \quad \text{Strömung kontrahiert} \end{cases}$$

Beispiel 3 Starre Rotation

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega(-y\vec{i} + x\vec{j})$$

$$P = -\omega y, \quad Q = \omega x$$



$$C: \ \vec{v} \perp \vec{n} \longrightarrow \cos\sphericalangle(\vec{v},\vec{n}) = 0$$

$$\vec{v} \cdot \vec{n} = v \cdot \cos\sphericalangle(\vec{v},\vec{n}) = 0$$

Fluß: $\oint_C (\vec{v} \cdot \vec{n})\, ds = 0$

Flächenintegral:

$$\frac{\partial P}{\partial x} = 0 = \frac{\partial Q}{\partial y}$$

$$\iint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = 0$$

Der Gaußsche Satz. Fluß und Divergenz

III. Verallgemeinerung des Gaußschen Satzes auf drei Dimensionen

- $\vec{\phi} = P\vec{i} + Q\vec{j} + R\vec{k}$, $\quad P = P(x,y,z)$, $\quad Q, R \ldots$

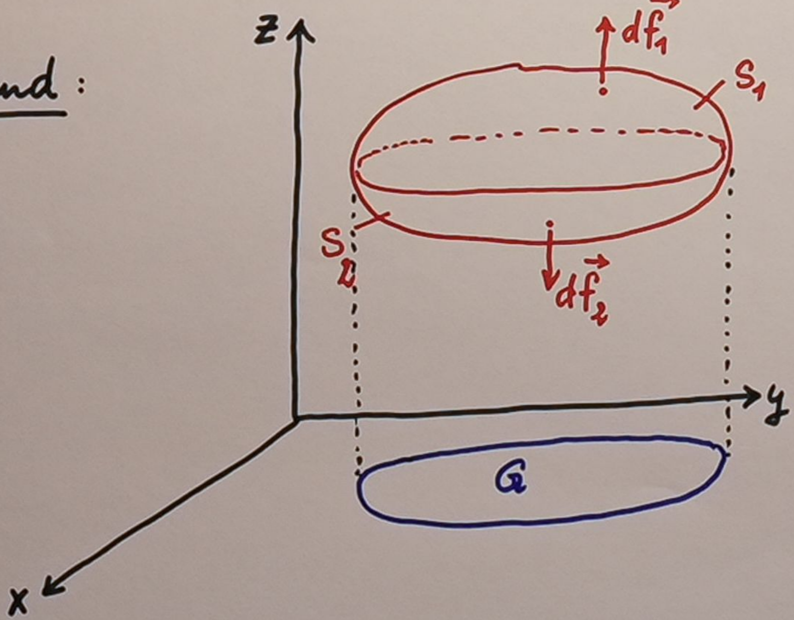
- $\oint_C (\vec{\phi}\cdot\vec{n})\,ds \longrightarrow \oiint_S (\vec{\phi}\cdot\vec{n})\,df = \oiint_S \vec{\phi}\cdot d\vec{f}$

$$= \oiint_S P\vec{i}\cdot d\vec{f} + \oiint_S Q\vec{j}\cdot d\vec{f} + \oiint_S R\vec{k}\cdot d\vec{f}$$

3. Summand:

S_1: $z_1(x,y)$

S_2: $z_2(x,y)$



$$\oiint_S R\vec{k}\cdot d\vec{f} = \iint_{S_1} R\vec{k}\cdot d\vec{f}_1 + \iint_{S_2} R\vec{k}\cdot d\vec{f}_2$$

$$d\vec{f}_1 = \left(-\frac{\partial z_1}{\partial x}\vec{i} - \frac{\partial z_1}{\partial y}\vec{j} + \vec{k}\right)dx\,dy$$

$$\longrightarrow \vec{k}\cdot d\vec{f}_1 = dx\,dy$$

$$d\vec{f}_2 = \left(\frac{\partial z_2}{\partial x}\vec{i} + \frac{\partial z_2}{\partial y}\vec{j} - \vec{k}\right)dx\,dy$$

$$\longrightarrow \vec{k}\cdot d\vec{f}_2 = -dx\,dy$$

$$\oiint_S R\vec{k}\cdot d\vec{f} = \iint_G \Big\{ R[x,y,z_1(x,y)] - R[x,y,z_2(x,y)] \Big\}\,dx\,dy$$

Andererseits:

$$\iiint_V \frac{\partial R}{\partial z}\, dx\,dy\,dz$$

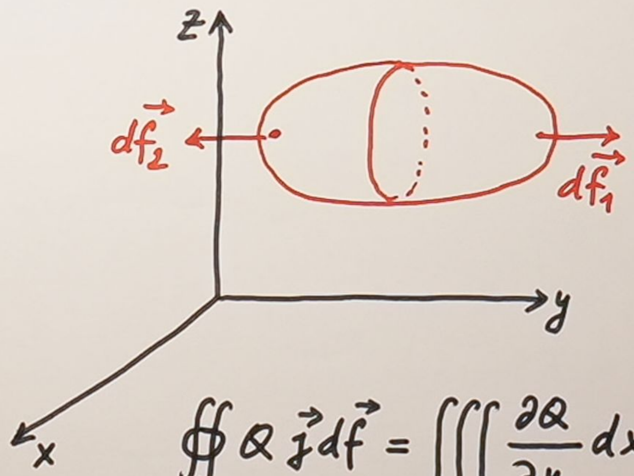
$$= \iint_G \left[\int_{z_2(x,y)}^{z_1(x,y)} \frac{\partial R}{\partial z}\, dz \right] dx\,dy$$

$$= \iint_G R(x,y,z)\Big|_{z=z_2(x,y)}^{z=z_1(x,y)} dx\,dy$$

$$= \iint_G \left\{ R\left[x,y,z_1(x,y)\right] - R\left[x,y,z_2(x,y)\right] \right\} dx\,dy$$

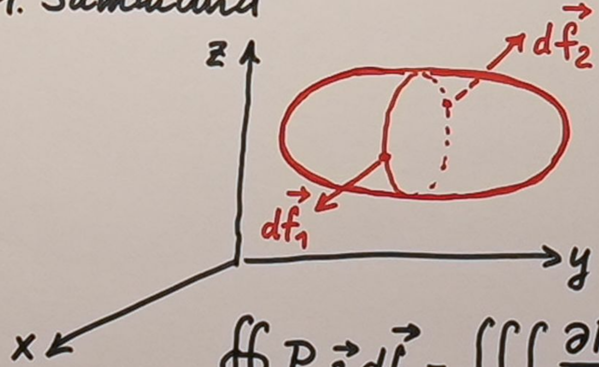
Vergleich: $\displaystyle\oiint_S R\,\vec{k}\,d\vec{f} = \iiint_V \frac{\partial R}{\partial z}\, dx\,dy\,dz$

2. Summand



$$\oiint_S Q\,\vec{j}\,d\vec{f} = \iiint_V \frac{\partial Q}{\partial y}\, dx\,dy\,dz$$

1. Summand



$$\oiint_S P\,\vec{i}\,d\vec{f} = \iiint_V \frac{\partial P}{\partial x}\, dx\,dy\,dz$$

zusammen:

$$\oiint_S \vec{\Phi}\,d\vec{f} = \iiint_V \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)}_{\operatorname{div}\vec{\Phi}} dx\,dy\,dz \quad , \quad \text{Gaußscher Satz}$$

Divergenz: $\operatorname{div}\vec{\phi} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \vec{\nabla}\cdot\vec{\phi}$

$$\oiint_S \vec{\phi}\, d\vec{f} = \iiint_V \operatorname{div}\vec{\phi}\cdot dV$$ Gaußscher Satz

Zur Bedeutung der Divergenz

Beispiel 1 Lineare Strömung

$\vec{v} = \alpha y \cdot \vec{i}\,, \quad \alpha > 0$

Fluß $\oint_C (\vec{v}\cdot\vec{n})\, ds = 0$

Divergenz $\operatorname{div}\vec{v} = 0$

Strömung inkompressibel

Beispiel 2 Radiale Strömung

$\vec{v} = \alpha\vec{r} = \alpha(x\vec{i} + y\vec{j})$

Fluß $\oint_C (\vec{v}\cdot\vec{n})\, ds = 2\pi\alpha R^2$

$\operatorname{div}\vec{v} = 2\alpha \begin{cases} > 0\,, & \text{Strömung expandiert, Quelle} \\ < 0\,, & \text{Strömung kontrahiert, Senke} \end{cases}$

wirbelfreies Quellenfeld

$\operatorname{div}\vec{v}$: Quellstärke

Beispiel 3 Starre Rotation

$\vec{v} = \vec{\omega}\times\vec{r} = \omega(-y\vec{i} + x\vec{j})$

Fluß $\oint_C (\vec{v}\cdot\vec{n})\, ds = 0$

Divergenz $\operatorname{div}\vec{v} = 0$

quellenfreies Wirbelfeld

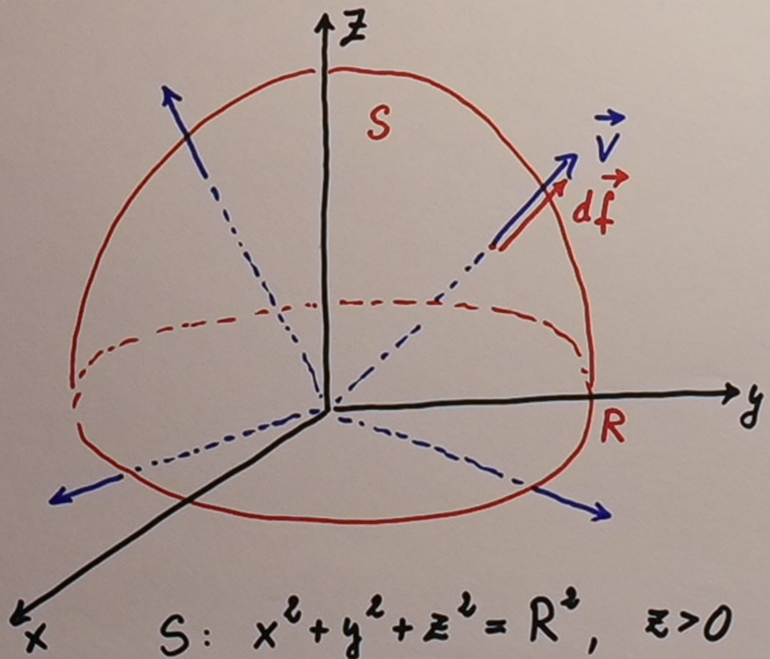
Gaußscher Satz. Fluß und Divergenz

IV. Beispiele

Beispiel 1: Radiale Strömung in drei Dimensionen

Strömungsfeld: $\vec{v} = \alpha \vec{r} = \alpha (x\vec{i} + y\vec{j} + z\vec{k})$

a) Fluß durch offene Fläche einer Halbkugel



$S: \; x^2 + y^2 + z^2 = R^2, \quad z > 0$

$d\vec{f} = \vec{e}_r \cdot r^2 \sin\vartheta \, d\vartheta \, d\varphi$

$\vec{v} = \alpha \cdot r \cdot \vec{e}_r$

$\vec{v} \cdot d\vec{f} = \alpha r^3 \sin\vartheta \, d\vartheta \, d\varphi$

auf S: $r = R$

Fluß: $\iint\limits_S \vec{v} \, d\vec{f}$

$= \alpha R^3 \int\limits_0^{\pi/2} \sin\vartheta \, d\vartheta \int\limits_0^{2\pi} d\varphi$

$= 2\pi \alpha R^3$

$= \frac{1}{2} \cdot \frac{4\pi}{3} R^3 \cdot 3\alpha$

b) Vollkugel (geschlossene Fläche)

Fluß:

$$\oiint\limits_S \vec{v} \, d\vec{f} = 2 \cdot 2\pi \alpha R^3 = \frac{4\pi}{3} R^3 \cdot 3\alpha$$

Divergenz: $\operatorname{div} \vec{v} = 3\alpha$, (= const)

$$\iiint\limits_V \operatorname{div} \vec{v} \, dV = 3\alpha \cdot \iiint\limits_{\text{Kugel}} dV$$

$$= \frac{4\pi}{3} R^3 \cdot 3\alpha$$

- $\operatorname{div} \vec{v} = \frac{1}{V} \oiint\limits_S \vec{v} \, d\vec{f}$

  „Flußdichte"

- 3 Dimensionen: $\operatorname{div} \vec{r} = 3$

Beispiel 2:

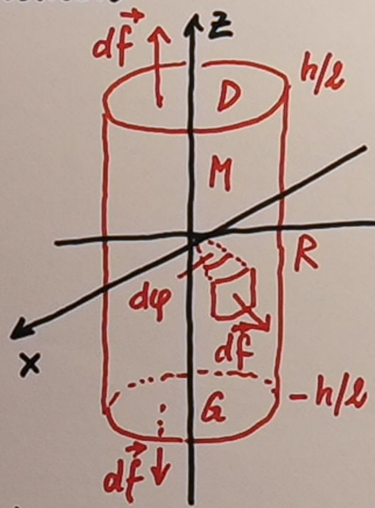
Vektorfeld: $\vec{\phi} = 3xy^2\cdot\vec{i} + 3x^2y\cdot\vec{j} + z^3\vec{k}$

$$\operatorname{div}\vec{\phi} = 3(x^2+y^2) + 3z^2$$

Volumen: Kreiszylinder

$$x^2+y^2 \le R^2$$

$$-\frac{h}{2} \le z \le \frac{h}{2}$$



Flächen:

$$G:\ \left.\begin{array}{l} x^2+y^2\le R^2,\ z=-\frac{h}{2} \\ d\vec{f} = -\vec{k}\cdot dx\,dy \end{array}\right\}\ \vec{\phi}\,d\vec{f} = \frac{1}{8}h^3\,dx\,dy$$

$$D:\ \left.\begin{array}{l} x^2+y^2\le R^2,\ z=\frac{h}{2} \\ d\vec{f} = \vec{k}\cdot dx\,dy \end{array}\right\}\ \vec{\phi}\,d\vec{f} = \frac{1}{8}h^3\,dx\,dy$$

$$M:\ \left.\begin{array}{l} x^2+y^2 = R^2,\ -\frac{h}{2}\le z\le\frac{h}{2} \\ d\vec{f} = (\cos\varphi\cdot\vec{i} + \sin\varphi\cdot\vec{j})\cdot R\,d\varphi\,dz \end{array}\right\}\ \vec{\phi}\,d\vec{f} = 6R^4\cos^2\varphi\,\sin^2\varphi\,d\varphi\,dz$$

Fluß: $\oiint_S \vec{\phi}\,d\vec{f}$, „$S = G + D + M$"

$$= \frac{1}{8}h^3\iint_G dx\,dy + 6R^4\int_0^{2\pi}\cos^2\varphi\,\sin^2\varphi\,d\varphi\int_{-h/2}^{h/2}dz$$

$$+ \frac{1}{8}h^3\iint_D dx\,dy$$

$$= \frac{1}{8}h^3\cdot\pi R^2 + 6R^4\cdot\frac{2\pi}{8}\cdot h + \frac{1}{8}h^3\pi R^2 = \frac{\pi}{2}R^2h\left(\frac{1}{2}h^2 + 3R^2\right)$$

Divergenz und Volumenintegral: $\operatorname{div}\vec{\phi} = 3(\varrho^2 + z^2)$

$$\iiint_V \operatorname{div}\vec{\phi}\,dV = 3\int_0^R \varrho\,d\varrho\int_0^{2\pi}d\varphi\int_{-h/2}^{h/2}dz\,(\varrho^2+z^2)$$

$$= 3\cdot 2\pi\cdot h\int_0^R\varrho^3\,d\varrho + 3\cdot 2\pi\int_0^R\varrho\,d\varrho\int_{-h/2}^{h/2}z^2\,dz$$

$$= \frac{\pi}{2}R^2h\left(3R^2 + \frac{1}{2}h^2\right)$$

Übereinstimmung

Beispiel 3:

speziell $\vec{a} = \mathrm{rot}\,\vec{\phi}$

$$\underbrace{\iiint_V \mathrm{div}\,\mathrm{rot}\,\vec{\phi}\, dV = \oiint_S}_{\text{Gauß}} \underbrace{\mathrm{rot}\,\vec{\phi}\, d\vec{f} = \oint_{C_S}}_{\text{Stokes}} \vec{\phi}\, d\vec{r} = 0$$

keine Randkurve für geschlossene Fläche S

$$\longrightarrow \quad \mathrm{div}\,\mathrm{rot}\,\vec{\phi} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{\phi}) = 0 .$$

explizit:

$$\mathrm{div}\,\mathrm{rot}\,\vec{\phi} = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}\right)$$
$$+ \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

Beispiel 4: Laplace-Operator und die Greenschen Identitäten

$$\vec{\phi} = \lambda \vec{A} \quad , \quad \lambda = \lambda(\vec{r})$$

$$\mathrm{div}\,\vec{\phi} = \mathrm{div}(\lambda\vec{A}) = \frac{\partial}{\partial x}(\lambda A_x) + \frac{\partial}{\partial y}(\lambda A_y) + \frac{\partial}{\partial z}(\lambda A_z)$$

$$\underline{\mathrm{div}(\lambda\vec{A}) = \lambda\cdot\mathrm{div}\,\vec{A} + \vec{A}\cdot\mathrm{grad}\,\lambda}$$

„Produktregel"

sei $\vec{A} = \mathrm{grad}\,U$

$$\mathrm{div}(\lambda\,\mathrm{grad}\,U) = \lambda\cdot\mathrm{div}(\mathrm{grad}\,U) + \mathrm{grad}\,U\cdot\mathrm{grad}\,\lambda$$

$$\mathrm{div}\,\mathrm{grad}\,U = \frac{\partial}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial}{\partial y}\frac{\partial U}{\partial y} + \frac{\partial}{\partial z}\frac{\partial U}{\partial z}$$
$$= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \equiv \Delta U$$

mit $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \vec{\nabla}\cdot\vec{\nabla}$ Laplace-Operator

$$\vec{\phi} = \lambda\,\mathrm{grad}\,U \xrightarrow{\text{Gauß}}$$

$$\oiint_S \lambda\,\mathrm{grad}\,U\, d\vec{f} = \iiint_V (\lambda\cdot\Delta U + \mathrm{grad}\,U\cdot\mathrm{grad}\,\lambda)\, dV$$

1. Greensche Identität

$\lambda \longleftrightarrow U$, Subtraktion

$$\oiint_S (U\,\mathrm{grad}\,\lambda - \lambda\,\mathrm{grad}\,U)\, d\vec{f} = \iiint_V (U\cdot\Delta\lambda - \lambda\cdot\Delta U)\, dV$$

2. Greensche Identität

Differentialoperatoren in krummlinigen Orthogonalkoordinaten

Wiederholung: ebene Polarkoordinaten

$$x = r\cos\varphi \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\varphi \qquad \varphi = \arctan\frac{y}{x}$$

Basis-Einheitsvektoren:

$$\vec{e}_r = \cos\varphi\cdot\vec{i} + \sin\varphi\cdot\vec{j}$$
$$\vec{e}_\varphi = -\sin\varphi\cdot\vec{i} + \cos\varphi\cdot\vec{j}$$

$$\vec{i} = \cos\varphi\cdot\vec{e}_r - \sin\varphi\cdot\vec{e}_\varphi$$
$$\vec{j} = \sin\varphi\cdot\vec{e}_r + \cos\varphi\cdot\vec{e}_\varphi$$

Vektorkomponenten:

$$\vec{A} = A_x\vec{i} + A_y\vec{j} = A_r\vec{e}_r + A_\varphi\vec{e}_\varphi$$

$$A_r = \vec{A}\cdot\vec{e}_r = A_x\cdot(\vec{i}\,\vec{e}_r) + A_y\cdot(\vec{j}\,\vec{e}_r)$$
$$= A_x\cdot\cos\varphi + A_y\cdot\sin\varphi$$
$$A_\varphi = -A_x\cdot\sin\varphi + A_y\cdot\cos\varphi$$

$$A_x = \vec{A}\cdot\vec{i} = A_r\cos\varphi - A_\varphi\sin\varphi$$
$$A_y = \vec{A}\cdot\vec{j} = A_r\sin\varphi + A_\varphi\cos\varphi$$

Vektorkomponenten transformieren sich wie Basis-Einheitsvektoren.

<u>Beispiel 1:</u> $\vec{A} = \operatorname{grad} U$

$$(\operatorname{grad} U)_r = \frac{\partial U}{\partial x}\cos\varphi + \frac{\partial U}{\partial y}\sin\varphi$$
$$= \left(\frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial\varphi}\frac{\partial\varphi}{\partial x}\right)\cos\varphi$$
$$+ \left(\frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial\varphi}\frac{\partial\varphi}{\partial y}\right)\sin\varphi$$

$$\left[\ \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\varphi \quad \vdots\ \right.$$

$$= \left(\frac{\partial U}{\partial r}\cos\varphi - \frac{\partial U}{\partial\varphi}\cdot\cancel{\frac{1}{r}}\sin\varphi\right)\cos\varphi$$
$$+ \left(\frac{\partial U}{\partial r}\sin\varphi + \frac{\partial U}{\partial\varphi}\cdot\cancel{\frac{1}{r}}\cos\varphi\right)\sin\varphi$$
$$= \frac{\partial U}{\partial r}$$

analog: $(\operatorname{grad} U)_\varphi = \frac{1}{r}\frac{\partial U}{\partial \varphi}$

$$\operatorname{grad} U = \frac{\partial U}{\partial r}\cdot \vec{e}_r + \frac{1}{r}\frac{\partial U}{\partial \varphi}\cdot \vec{e}_\varphi$$

Beispiel 2: $\operatorname{div} \vec{A}$

$$\operatorname{div}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}$$

$$= \left(\frac{\partial A_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial A_x}{\partial \varphi}\frac{\partial \varphi}{\partial x}\right) + \left(\frac{\partial A_y}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial A_y}{\partial \varphi}\frac{\partial \varphi}{\partial y}\right)$$

$$\operatorname{div}\vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi}$$

Warnung: Nabla-Operator

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\varphi \cdot \frac{1}{r}\frac{\partial}{\partial \varphi} \quad \rightarrow \vec{\nabla} U = \operatorname{grad} U$$

$$\vec{\nabla}\cdot\vec{A} = \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\varphi \cdot \frac{1}{r}\frac{\partial}{\partial \varphi}\right)\left(A_r \vec{e}_r + A_\varphi \vec{e}_\varphi\right)$$

$$\stackrel{?}{=} \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi}$$ falsch!

$$\frac{d\vec{e}_r}{d\varphi} = \vec{e}_\varphi, \quad \frac{d\vec{e}_\varphi}{d\varphi} = -\vec{e}_r$$

$$\vec{\nabla}\cdot\vec{A} = \vec{e}_r \frac{\partial}{\partial r}(A_r \vec{e}_r) + \vec{e}_r \frac{\partial}{\partial r}(A_\varphi \vec{e}_\varphi) + \vec{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}(A_r \vec{e}_r) + \vec{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}(A_\varphi \vec{e}_\varphi)$$

$$= \vec{e}_r^{\,2}\frac{\partial A_r}{\partial r} + \cancel{\vec{e}_r \vec{e}_\varphi \frac{\partial A_\varphi}{\partial r}} + \cancel{\vec{e}_\varphi \vec{e}_r \cdot \frac{1}{r}\frac{\partial A_r}{\partial \varphi}} + \vec{e}_\varphi \frac{1}{r} A_r \vec{e}_\varphi + \vec{e}_\varphi \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi}\vec{e}_\varphi - \cancel{\vec{e}_\varphi \frac{1}{r} A_\varphi \vec{e}_r}$$

$$= \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} \quad ✓$$

Beispiel 3: ΔU

$$\Delta U = \operatorname{div} \operatorname{grad} U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}$$

Zusammenfassung und Formelsammlung

Linienelement: $ds^2 = g_1^2 dx_1^2 + g_2^2 dx_2^2 + g_3^2 dx_3^2$

	x_1	x_2	x_3	g_1	g_2	g_3
kartesisch	x	y	z	1	1	1
Zylinder	ϱ	φ	z	1	ϱ	1
Kugel	r	ϑ	φ	1	r	$r\sin\vartheta$

Volumenelement: $dV = g_1 g_2 g_3 \cdot dx_1 \cdot dx_2 \cdot dx_3$

Basis-Einheitsvektoren: $\vec{e}_1 = \frac{1}{g_1}\frac{\partial \vec{r}}{\partial x_1}$, $\vec{e}_2 = \frac{1}{g_2}\frac{\partial \vec{r}}{\partial x_2}$

$\vec{e}_3 = \frac{1}{g_3}\frac{\partial \vec{r}}{\partial x_3}$

Gradient: $\operatorname{grad} u = \frac{1}{g_1}\frac{\partial u}{\partial x_1}\vec{e}_1 + \frac{1}{g_2}\frac{\partial u}{\partial x_2}\vec{e}_2 + \frac{1}{g_3}\frac{\partial u}{\partial x_3}\vec{e}_3$

Nabla: $\vec{\nabla} = \vec{e}_1 \frac{1}{g_1}\frac{\partial}{\partial x_1} + \vec{e}_2 \frac{1}{g_2}\frac{\partial}{\partial x_2} + \vec{e}_3 \frac{1}{g_3}\frac{\partial}{\partial x_3}$

Divergenz:

$$\operatorname{div}\vec{A} = \frac{1}{g_1 g_2 g_3}\left[\frac{\partial}{\partial x_1}(A_1 g_2 g_3) + \frac{\partial}{\partial x_2}(g_1 A_2 g_3) + \frac{\partial}{\partial x_3}(g_1 g_2 A_3)\right]$$

Rotation:

$$\operatorname{rot}\vec{A} = \frac{1}{g_1 g_2 g_3}\begin{vmatrix} g_1\vec{e}_1 & g_2\vec{e}_2 & g_3\vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ g_1 A_1 & g_2 A_2 & g_3 A_3 \end{vmatrix}$$

Beispiel: Kugelkoordinaten

$$(\operatorname{rot}\vec{A})_r = \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(\sin\vartheta A_\varphi) - \frac{\partial A_\vartheta}{\partial\varphi}\right]$$

Laplace-Operator:

$$\Delta u = \frac{1}{g_1 g_2 g_3}\left[\frac{\partial}{\partial x_1}\left(\frac{g_2 g_3}{g_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{g_1 g_3}{g_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{g_1 g_2}{g_3}\frac{\partial u}{\partial x_3}\right)\right]$$